Structural Analysis

Lecture Series

SA58: Analysis of a Continuous Tapered Beam

Slope-Deflection Method

This document is a written version of video lecture SA58, which can be found online at the web addresses listed below

Educative Technologies, LLC

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2018

Structural Analysis – SA58

Analysis of a Continuous Tapered Beam using the Slope-deflection Method

Consider the continuous two-span railroad bridge shown below. It is a non-prismatic beam with a deep section at point B that tapers to a shallower depth at ends A and C.

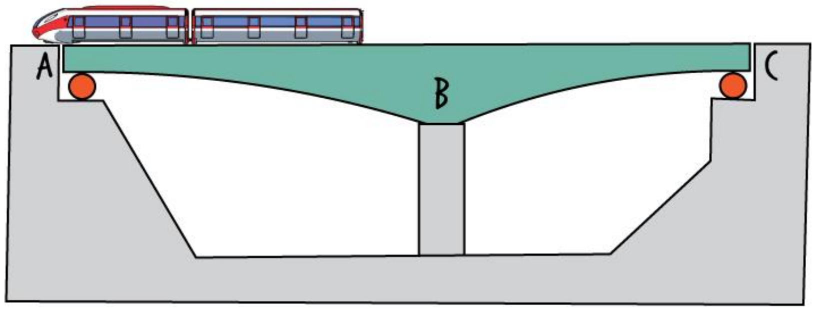


Figure 1: A continuous tapered beam

Since the cross-section of the beam does not have a constant moment of inertia (I), the standard slope-deflection equations do not apply. We need a revised formulation that takes the variable moment of inertia into account for modeling and solving the problem.

This document consists of two main parts. In Part 1, we develop the revised slope-deflection formulation. In Part 2, we apply the formulation to analyze the non-prismatic beam shown in Figure 1.

Part 1: Revised Slope-Deflection Formulation

To make the derivation and the resulting equations manageable, we start by defining the ratio of the end heights of a tapered beam. The height at end A is assumed to be h_0 and the height at the right end of the beam, at point B, is set to $4h_0$, as shown in Figure 2.

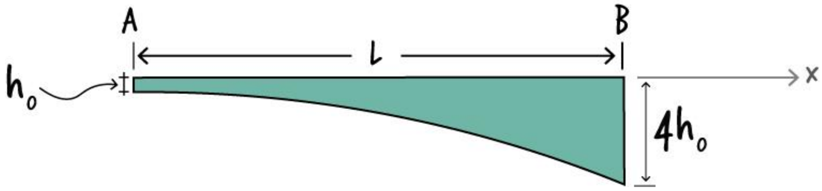


Figure 2: Beam with a variable moment of inertia

Given the nonlinear shape of the beam, its height can be represented using a quadratic equation, like this: $h(x)=a+bx+cx^2$ where coefficients a, b, and c are to be determined using three boundary conditions. At $x = 0$, the function should return h_0; at $x = L$, the height is $4h_0$; and the slope of the curve at $x = 0$ is zero. Writing these conditions in algebraic form, we get:

$$h(0)=a+b(0)+c(0)^2=h_0 \quad [1]$$

$$h(L)=a+b(L)+c(L)^2=4h_0 \quad [2]$$

$$h'(0)=b+2c(0)=0 \quad [3]$$

Solving the above equations for the unknown coefficients results in:

$$a=h_0, b=0, c=\frac{3h_0}{L^2}$$

Substituting these values in the expression for $h(x)$, the height function becomes:

$$h(x)=h_0+3h_0\left(\frac{x}{L}\right)^2 \quad [4]$$

Let us assume a rectangular cross-section for the beam. Then, we can express the moment of inertia in terms of $h(x)$, like this:

$$f(x)=\frac{1}{12}bh^3(x) \quad [5]$$

Substituting Equation [4] in [5], we get:

$$f(x)=\frac{bh_0^3}{12}\left(1+3\frac{x^2}{L^2}\right)^3 \quad [6]$$

Using $I_0=\frac{bh_0^3}{12}$ as the moment of inertia of the beam at end A, $f(x)$ can be written as:

$$f(x)=I_0\left(1+3\frac{x^2}{L^2}\right)^3 \quad [7]$$

Now that we have the moment of inertia expressed as a function of x, we are ready to proceed with the revised slope-deflection formulation.

Let's label the end rotations of the beam as θ_A and θ_B, and refer to the end moments as M_{AB} and M_{BA}, as shown in Figure 3. Note that all four parameters are shown in the counterclockwise (positive) direction. The beam is assumed to be simply supported.

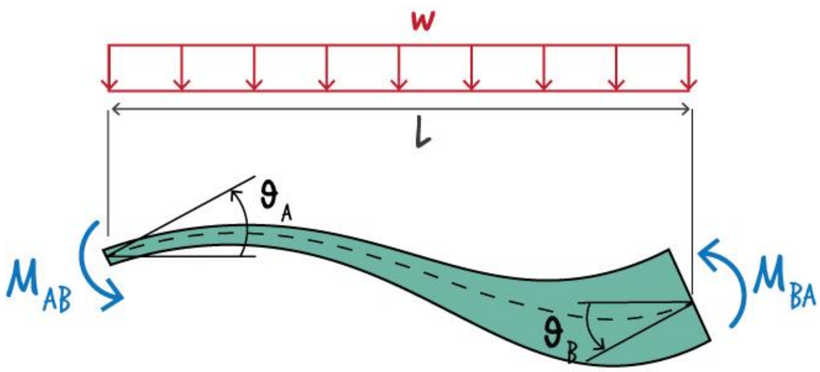
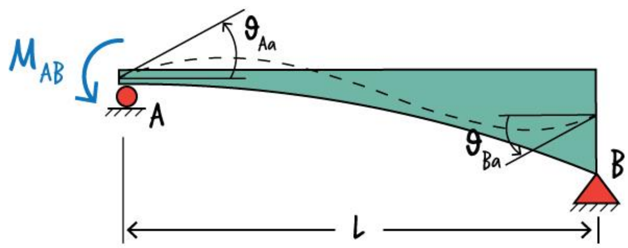


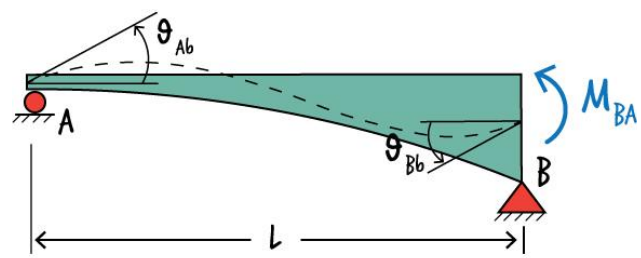
Figure 3: Positive end rotations and moments in a beam

We intend to write the slope-deflection equations for this tapered beam. This means we wish to express the end moments (M_{AB} and M_{BA}) in terms of the end rotations (θ_A and θ_B), w, L, and section and material properties. To do so, we are going to use the principle of superposition coupled with the method of virtual work.

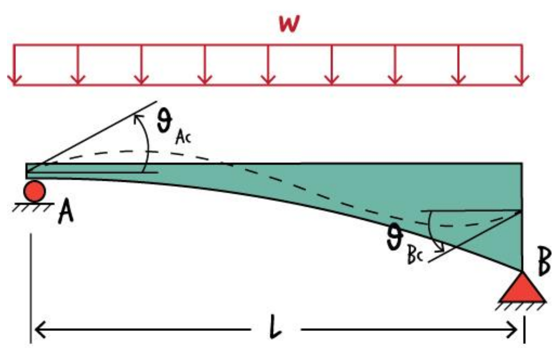
To start, we need to write θ_A in terms of M_{AB}, M_{BA} and w. We can determine this relationship by placing M_{AB}, M_{BA}, and w on the beam, as applied loads, and calculate the end rotation that each causes. Figures [4a], [4b], and [4c] show the beam subjected to each load.



(a) *Beam subjected to* M_{AB}



(b) *Beam subjected to* M_{BA}



(c) *Beam subjected to distributed load* w

Figure 4: Loads applied to a tapered beam decoupled

We can determine these end rotations, referred to as ϑ_{Aa}, ϑ_{Ab}, and ϑ_{Ac} in the figure, using the virtual work method. Then, using the principle of superposition, we can calculate the total end rotation this way:

$$\vartheta_A = \vartheta_{Aa} + \vartheta_{Ab} + \vartheta_{Ac} \qquad [8]$$

Similarly, after computing ϑ_{Ba} , ϑ_{Bb} and ϑ_{Bc} , we can determine the total rotation at B using the following equation:

$$\vartheta_B = \vartheta_{Ba} + \vartheta_{Bb} + \vartheta_{Bc} \qquad [9]$$

According to the virtual work method, the rotation at end A due to M_{AB}, shown in Figure [4a], can be expressed as:

$$\vartheta_{Aa} = \int_0^L \frac{M(x)m(x)}{EI}dx \qquad [10]$$

Where $M(x)$ is the bending moment equation due to the applied load, here M_{AB}, and $m(x)$ is the bending moment equation due to a virtual unit moment placed in the direction of ϑ_{Aa} . Please keep in mind that the moment of inertia (I) is defined using algebraic function $f(x)$ per Equation [7].

The free-body diagram of the beam subjected to a unit moment in the direction of M_{AB} is shown in Figure 5.

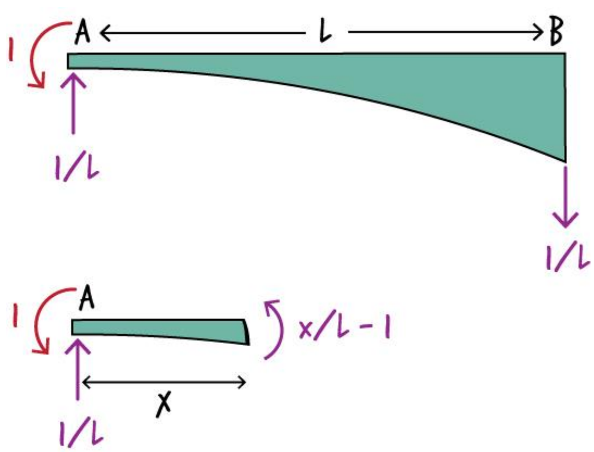


Figure 5: Tapered beam subjected to a unit counterclockwise moment at A

Note the bending moment expression in the figure; it is: $(x/L) - 1$. If we multiply it by M_{AB}, we get the expression for $M(x)$, as shown below.

$$M(x) = M_{AB}\left(\frac{x}{L} - 1\right) \qquad [11]$$

In Equation [10], $m(x)$ refers to bending moment equation for the beam due to a unit moment placed in the direction of ϑ_{Aa}. We have already determined this moment (see Figure 6), it is: $m(x) = (x/L) - 1$. Therefore, Equation [10] can be written as:

$$\vartheta_{Aa} = \int_0^L \frac{M_{AB}\left(\frac{x}{L} - 1\right)\left(\frac{x}{L} - 1\right)}{EI} dx \qquad [12]$$

Or,

$$\vartheta_{Aa} = \int_0^L \frac{M_{AB}\left(\frac{x}{L} - 1\right)\left(\frac{x}{L} - 1\right)}{EI_0\left(1 + 3\frac{x^2}{L^2}\right)^3} dx \qquad [13]$$

Or,

$$\vartheta_{Aa} = \frac{M_{AB}}{EI_0} \int_0^L \frac{\left(\frac{x}{L} - 1\right)\left(\frac{x}{L} - 1\right)}{\left(1 + 3\frac{x^2}{L^2}\right)^3} dx \qquad [14]$$

To facilitate the required integration in the above equation, let's make a substitution. Let $\eta = x/L$ where $0 \le \eta \le 1$. Hence, Equation [14] becomes:

$$\vartheta_{Aa} = \frac{M_{AB}}{EI_0} \int_0^1 \frac{(\eta - 1)(\eta - 1)}{(1 + 3\eta^2)^3} dx \qquad [15]$$

And since $x = \eta L$, then $dx = L\, d\eta$. Or,

$$\theta_{Aa} = \frac{M_{AB}}{EI_0} L \int_0^1 \frac{(\eta-1)(\eta-1)}{(1+3\eta^2)^3} d\eta \qquad [16]$$

The above integral expression evaluates to 0.21025. Therefore,

$$\theta_{Aa} = 0.21025 \frac{M_{AB} L}{EI_0} \qquad [17]$$

Now, let's turn our attention to θ_{Ba}. The virtual work method tells us that this rotation can be computed using:

$$\theta_{Ba} = \int_0^L \frac{M(x)m(x)}{EI} dx \qquad [18]$$

Where $M(x)$ is given by Equation [11], and $m(x)$ is the bending moment equation due to a unit load placed at B in the direction of θ_{Ba}. According to the free-body diagram shown in Figure 6, $m(x) = x/L$.

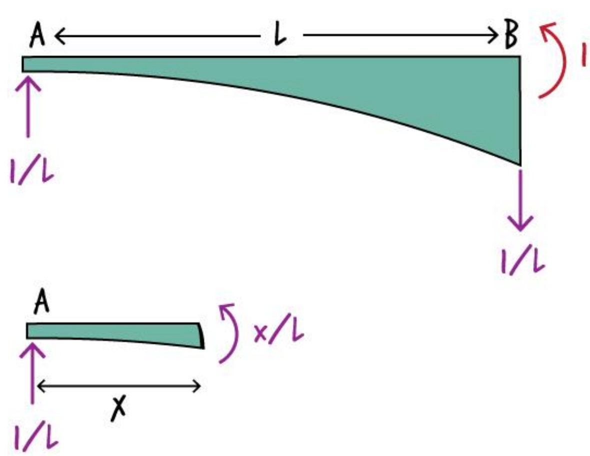


Figure 6: Tapered beam subjected to a unit counterclockwise moment at B

Therefore, we can expand Equation [18] as follows:

Or,

$$\vartheta_{Ba} = \frac{M_{AB}}{EI_0}\int_0^L \frac{(\frac{x}{L}-1)(\frac{x}{L})}{(1+3\frac{x^2}{L^2})^3}dx \quad [19]$$

Making the same substitution as before ($\eta = x/L$), we can write:

$$\vartheta_{Ba} = \frac{M_{AB}}{EI_0}L\int_0^1 \frac{(\eta-1)(\eta)}{(1+3\eta^2)^3}d\eta \quad [20]$$

Or,

$$\vartheta_{Ba} = -0.047725\frac{M_{AB}L}{EI_0} \quad [21]$$

We are now finished with the loading case depicted in Figure [4a].

Now, let's consider the second loading case, the one shown in Figure [4b]. To determine ϑ_{Ab}, using the virtual work method, we need to write the moment equation for the beam due to M_{BA}. The moment equation due to a unit moment applied at point B, in the direction of M_{BA}, is given in Figure 7; it is x/L. Therefore:

$$M(x) = \frac{x}{L}M_{BA} \quad [22]$$

The moment equation due to a virtual unit load placed in the direction of ϑ_{Ab} is shown in Figure 5. That is, $m(x) = x/L - 1$. Then, we can write:

$$\vartheta_{Ab} = \int_0^L \frac{M_{BA}(\frac{x}{L})(\frac{x}{L}-1)}{EI}dx \quad [23]$$

Or,

$$\vartheta_{Ab} = \frac{M_{BA}}{EI_0} \int_0^L \frac{(\frac{x}{L})(\frac{x}{L}-1)}{(1+3\frac{x^2}{L^2})^3} dx \qquad [24]$$

Or,

$$\vartheta_{Ab} = \frac{M_{BA}}{EI_0} L \int_0^1 \frac{(\eta)(\eta-1)}{(1+3\eta^2)^3} d\eta \qquad [25]$$

The above equation evaluates to:

$$\vartheta_{Ab} = -0.047725 \frac{M_{BA} L}{EI_0} \qquad [26]$$

To determine ϑ_{Bb}, we use the same $M(x)$ as above, but need to determine $m(x)$, the moment equation due to a virtual unit load placed in the direction of the rotation at B. The moment is given in Figure 6; it is $m(x) = x/L$. Consequently, we can write:

$$\vartheta_{Bb} = \int_0^L \frac{M_{BA}(\frac{x}{L})(\frac{x}{L})}{EI} dx \qquad [27]$$

Or,

$$\vartheta_{Bb} = \frac{M_{BA}}{EI_0} \int_0^L \frac{(\frac{x}{L})(\frac{x}{L})}{(1+3\frac{x^2}{L^2})^3} dx \qquad [28]$$

Or,

$$\vartheta_{Bb} = \frac{M_{BA}}{EI_0} L \int_0^1 \frac{(\eta)(\eta)}{(1+3\eta^2)^3} d\eta \qquad [29]$$

Therefore,

$$\vartheta_{Bb} = 0.030400\frac{M_{BA}L}{EI_0} \qquad [30]$$

The last loading case to consider is the one shown in Figure [4c]. The moment equation due to the applied load can be written based on the free-body diagram drawn in Figure 7. The moment equation is:

$$M(x) = \frac{wL}{2}x - \frac{wx^2}{2} \qquad [31]$$

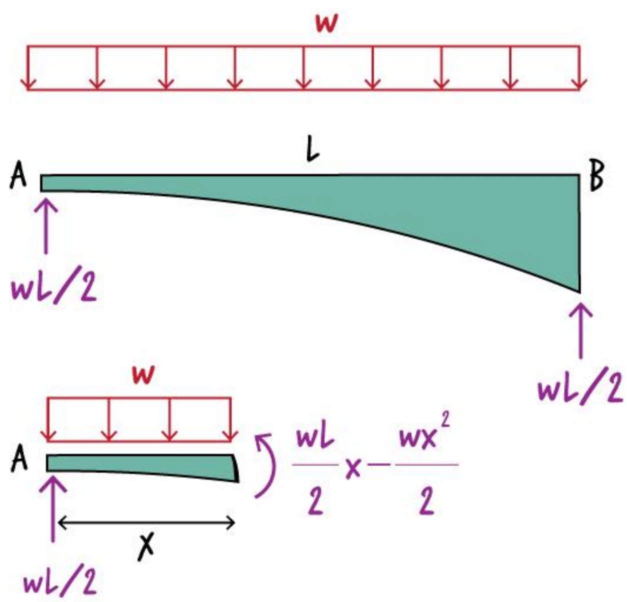


Figure 7: Tapered beam subjected to a uniformly distributed load

To calculate rotation ϑ_{Ac}, we need $m(x)$, the bending moment equation, due to a unit virtual moment placed at A. $m(x) = x/L - 1$ (see Figure 5 for details). So, the rotation we are after can be written as:

$$\vartheta_{Ac} = \int_0^L \frac{M(x)m(x)}{EI}dx = \frac{W}{2EI_0}\int_0^L \frac{(Lx - x^2)(\frac{x}{L} - 1)}{(1 + 3\frac{x^2}{L^2})^3}dx \qquad [32]$$

Substituting η for x/L, and ηL for x, we get:

$$\vartheta_{Ac} = \frac{W}{2EI_0} L \int_0^1 \frac{(\eta L^2 - \eta^2 L^2)(\eta - 1)}{(1+3\eta^2)^3} d\eta \qquad [33]$$

Or,

$$\vartheta_{Ac} = \frac{W}{2EI_0} L^3 \int_0^1 \frac{(\eta - \eta^2)(\eta - 1)}{(1+3\eta^2)^3} d\eta \qquad [34]$$

The above equation evaluates to:

$$\vartheta_{Ac} = -0.016475 \frac{WL^3}{EI_0} \qquad [35]$$

Finally, we can determine ϑ_{Bc} as follows.

$$\vartheta_{Bc} = \frac{W}{2EI_0} L^3 \int_0^1 \frac{(\eta - \eta^2)(\eta)}{(1+3\eta^2)^3} d\eta \qquad [36]$$

Or,

$$\vartheta_{Bc} = 0.0073875 \frac{WL^3}{EI_0} \qquad [37]$$

Now, we can write Equations [8] and [9] in expanded form as:

$$\vartheta_A = 0.21025 \frac{M_{AB} L}{EI_0} - 0.047725 \frac{M_{BA} L}{EI_0} - 0.016475 \frac{WL^3}{EI_0} \qquad [38]$$

$$\vartheta_B = -0.047725\frac{M_{AB}L}{EI_0} + 0.030400\frac{M_{BA}L}{EI_0} + 0.0073875\frac{WL^3}{EI_0} \quad [39]$$

Solving Equations [38] and [39] simultaneously for M_{AB} and M_{BA}, we get:

$$M_{AB} = \frac{EI_0}{L}(7.38955\,\vartheta_A + 11.6009\,\vartheta_B) + 0.036042\,wL^2 \quad [40]$$

$$M_{BA} = \frac{EI_0}{L}(11.6009\,\vartheta_A + 51.10700\,\vartheta_B) - 0.18643\,wL^2 \quad [41]$$

These are the revised slope-deflection equations for the tapered beam with a rectangular cross-section as shown in Figure 8.

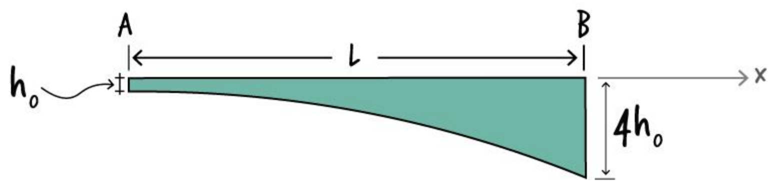


Figure 8: A shallow to deep tapered beam

But, what if the beam is deeper at the left end than at the right end? What would be the slope-deflection equations for the beam shown in Figure 9?

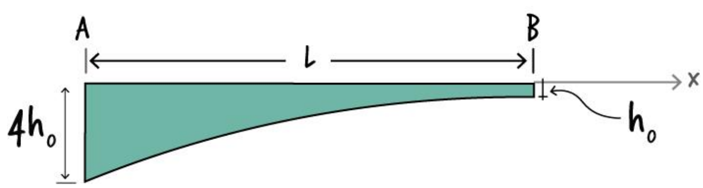


Figure 9: A deep to shallow tapered beam

The difference between the two beams is in function $h(x)$. For the beam in Figure [8], height is defined using Equation [4]. For the beam in Figure [9], however, the height function is:

$$h(x) = h_0\left(4 - 6\frac{x}{L} + 3\frac{x^2}{L^2}\right)^3 \quad [42]$$

We can obtain the slope-deflection equations for the beam in Figure [9] either by derivation or directly from Equations [40] and [41] by switching the position of the numerical coefficients, as done below.

$$M_{AB} = \frac{EI_0}{L}(51.10700\,\theta_A + 11.6009\,\theta_B) + 0.18643\,wL^2 \qquad [43]$$

$$M_{BA} = \frac{EI_0}{L}(11.6009\,\theta_A + 7.38955\,\theta_B) - 0.036042\,wL^2 \qquad [44]$$

Having the two sets of slope-deflection equations, we are now ready to analyze the bridge.

Part 2: Bridge Analysis

Suppose the left span of the bridge is 10 meters long and the right span is 8 meters long. We wish to analyze the continuous beam under a uniformly distributed load of 20 kN/m placed on the left span only, as shown below.

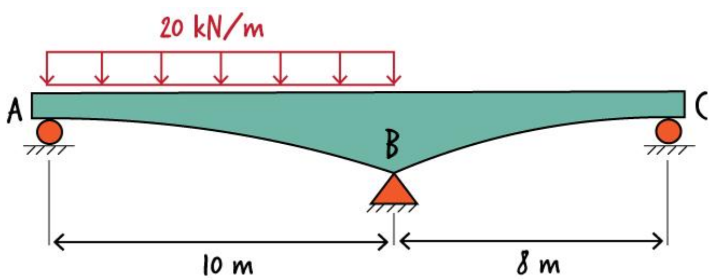


Figure 10: A two-span tapered beam subjected to a uniformly distributed load

According to the slope-deflection method, we need to write the slope-deflection equations for each span. For member AB, we use Equations [40] and [41], yielding:

$$M_{AB} = \frac{EI_0}{10}(7.38955\,\theta_A + 11.6009\,\theta_B) + 0.036042(20)(10)^2 \qquad [45]$$

$$M_{BA} = \frac{EI_0}{10}(11.6009\,\theta_A + 51.10700\,\theta_B) - 0.18643(20)(10)^2 \qquad [46]$$

For segment BC, using Equations [43] and [44], we can write:

$$M_{BC} = \frac{EI_0}{8}(51.10700\,\vartheta_B + 11.6009\,\vartheta_C) \quad [47]$$

$$M_{CB} = \frac{EI_0}{8}(11.6009\,\vartheta_B + 7.38955\,\vartheta_C) \quad [48]$$

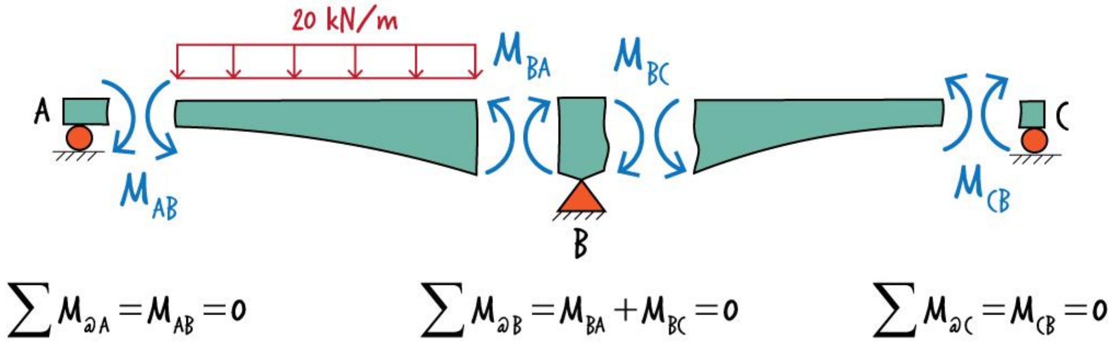


Figure 11: The joint free-body diagrams for the tapered beam

Then, we write the joint equilibrium equations, per the free-body diagram shown in Figure 11. The resulting equations, in simplified form, are:

$$\sum M_{@A} = \frac{EI_0}{10}(7.38955\,\vartheta_A + 11.6009\,\vartheta_B) + 72.83 = 0 \quad [49]$$

$$\sum M_{@B} = EI_0(1.16009\,\vartheta_A + 11.49919\,\vartheta_B + 1.45011\,\vartheta_C) - 372.856 = 0 \quad [50]$$

$$\sum M_{@C} = \frac{EI_0}{8}(11.6009\,\vartheta_B + 7.38955\,\vartheta_C) = 0 \quad [51]$$

Solving Equations [49] through [51] for the unknown joint rotations, we get:

$$\vartheta_A = -200.637 / EI_0 \quad [52]$$

$$\vartheta_B = 65.6671 EI_0 \quad [53]$$

$$\vartheta_C = -103.0897 EI_0 \quad [54]$$

Finally, substituting Equations [52] through [54] in the slope-deflection equations, we get the member end moments for AB and BC, as indicated below.

Equation [40] gives us: $M_{AB} = 0$. From Equation [41], we get: $M_{BA} = -270 \text{ kN.m}$. And, Equations [43] and [44], respectively, yield: $M_{BC} = 270 \text{ kN.m}$ and $M_{CB} = 0$.

Knowing the member-end moments, we can easily determine the member end shear forces using the static equilibrium equations, as shown in Figure 12.

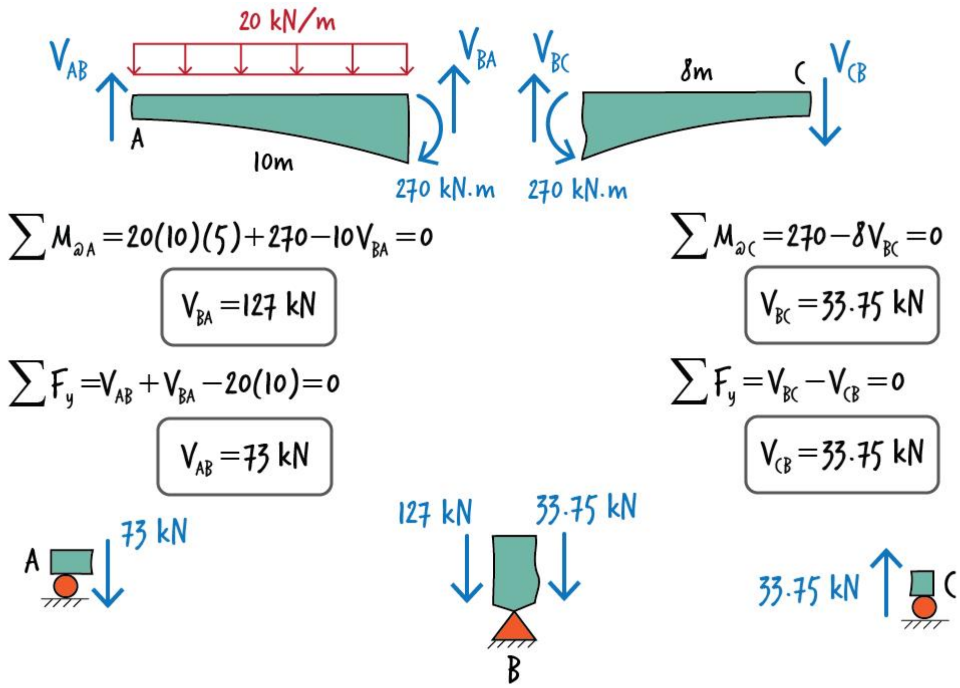


Figure 12: Shear force calculation for the tapered beam

Hence, the support reactions for the bridge are as shown in the diagram below.

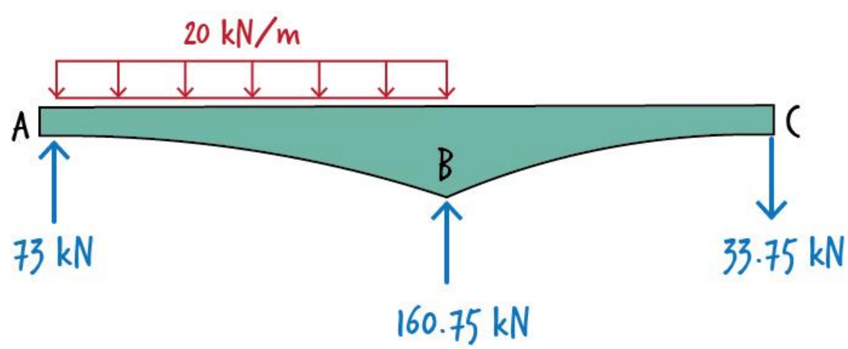


Figure 13: Support reactions for the tapered beam