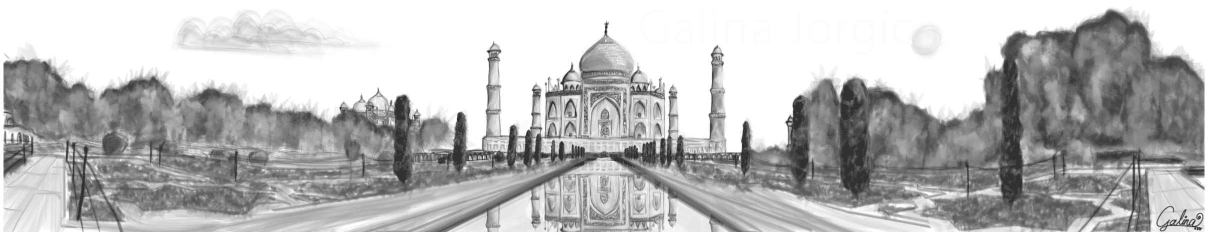
Structural Analysis

Lecture Series



SA60: Three-Moment Equation for Analyzing Continuous Beams: Part I

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2019

Structural Analysis – SA60

The three-moment equation for analyzing continuous beam

The three-moment equation is a single algebraic expression that relates the moment values at three consecutive points in a beam. We can use this equation for the analysis of continuous beams.

For example, consider the two-span continuous beam shown below.

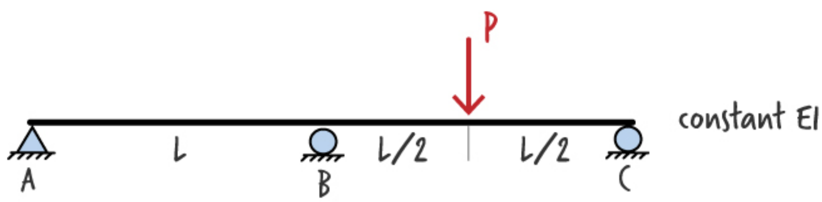


Figure 1: A two-span beam

Let's refer to the internal moments at points A, B, and C as M_A, M_B, and M_C, respectively. These moments are shown in Figure 2.

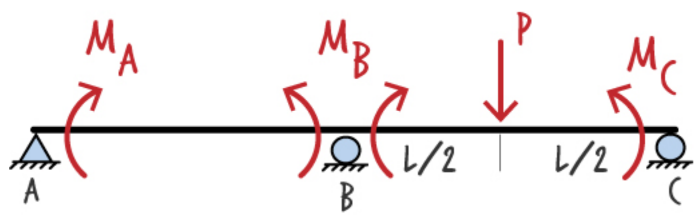


Figure 2: Bending moment at three consecutive points in a beam

The three-moment equation for the beam can be written as:

$$M_A + 4M_B + M_C = -\frac{3PL}{8} \quad [1]$$

In this case, since there is a pin at A and a roller at C, $M_A = M_C = 0$. Therefore, the equation can be simplified as shown below.

$$0 + 4M_B + 0 = -\frac{3PL}{8} \quad [2]$$

Solving Equation [2] for the unknown moment, we get: $M_B = -3PL/32$. With M_B known, the beam becomes statically determinate, which can then be analyzed using the static equilibrium equations.

Before we find out how the three-moment equation can be obtained, let's look at another example. Consider the three-span beam shown in Figure 3.

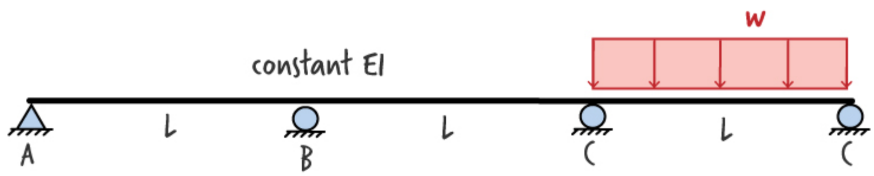


Figure 3: A continuous beam with three spans

The internal moments at four consecutive points (A, B, C, and D) are labeled as depicted in Figure 4.

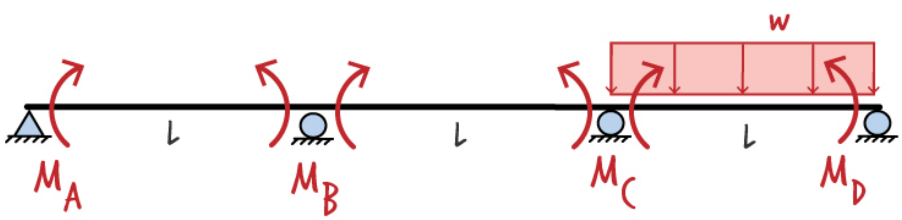


Figure 4: Four consecutive bending moments in a beam

In this case, we need to write two three-moment equations. We need one equation for the moments at A, B and C, and another equation for the moments at B, C, and D. The first three-moment equation can be written as:

$$M_A + 4M_B + M_C = 0 \quad [3]$$

The second three-moment equation is:

$$M_B + 4M_C + M_D = -\frac{wL^2}{4} \quad [4]$$

And since $M_A = M_D = 0$, we can simplify Equations [3] and [4] as follows:

$$4M_B + M_C = 0 \quad [5]$$

$$M_B + 4M_C = -\frac{wL^2}{4} \quad [6]$$

Solving Equations [5] and [6] for the unknown moments, we get: $M_B = wL^2/60$ and $M_C = -wL^2/15$. Knowing M_B and M_C, we can now analyze the beam using the static equilibrium equations.

Now let's derive the three-moment equation in its general form. Consider the generic two-span beam shown in Figure 5.

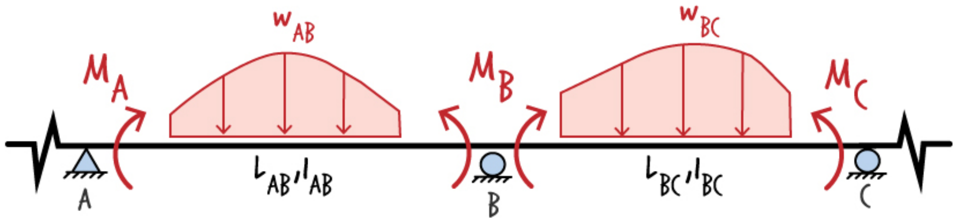


Figure 5: A two-span beam subjected to generalized loads

Here we are assuming that the two beam segments are subjected to different loads, and have different lengths and moments of inertia.

We can derive the three-moment equation using the slope-deflection formulation. If you are not familiar with the slope-deflection method, please review Lectures SA27 through SA33.

For a typical beam segment, such as segment AB in Figure 5, the slope-deflection equations can be written as follows:

$$M_{AB} = \frac{2EI_{AB}}{L_{AB}}(2\theta_A + \theta_B) + \Omega_{AB} \quad [7]$$

$$M_{BA} = \frac{2EI_{AB}}{L_{AB}}(\theta_A + 2\theta_B) - \Omega_{BA} \quad [8]$$

The last term in each equation above, denoted by the symbol Ω, is called the fixed-end moment, and can be easily determined using the applied loads. For example, if a beam of length L is subjected to a uniformly distributed load of w, the fixed-end moment is $wL^2/12$. If the beam is subjected to a concentrated load of P at its mid-point, $\Omega = PL/8$. Fixed-end moment values for various loads are tabulated in most structural analysis textbooks.

The sign convention used in the slope-deflection equations is shown in Figure 6.

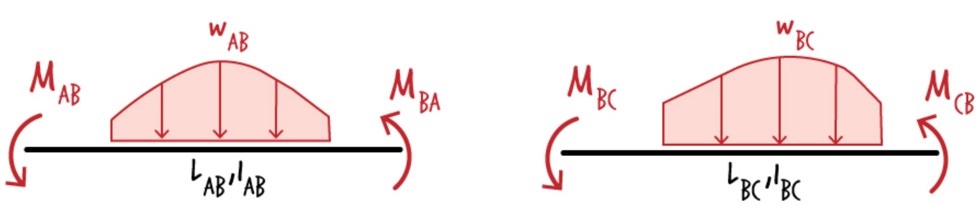


Figure 6: Member-end moments in beam segments shown using the slope-deflection notation

According to our slope-deflection formulation, counterclockwise moments and rotations are considered positive.

The slope-deflection equations for segment BC can be written in a similar manner.

$$M_{BC} = \frac{2EI_{BC}}{L_{BC}}(2\theta_B + \theta_C) + \Omega_{BC} \qquad [9]$$

$$M_{CB} = \frac{2EI_{BC}}{L_{BC}}(\theta_B + 2\theta_C) - \Omega_{CB} \qquad [10]$$

Comparing the diagrams shown in Figures 5 and 6, we can see that $M_{AB} = -M_A$. Using Equation [7], we can then write:

$$\frac{2EI_{AB}}{L_{AB}}(2\theta_A + \theta_B) + \Omega_{AB} = -M_A \qquad [11]$$

Also note that $M_{BA} = M_B$ and $M_{BC} = -M_B$ (see Figures 5 and 6).

Since the sum of the moments at point B must be zero, we can write:

$$M_{BA} + M_{BC} = 0 \qquad [12]$$

Using Equations [8] and [9], Equation [12] can be rewritten as follows:

$$\frac{2EI_{AB}}{L_{AB}}(\theta_A + 2\theta_B) - \Omega_{BA} + \frac{2EI_{BC}}{L_{BC}}(2\theta_B + \theta_C) + \Omega_{BC} = 0 \qquad [13]$$

Furthermore, comparing the Figures 5 and 6 at point C, we can see that $M_{CB} = M_C$. Then, using Equation [10] we can write:

$$\frac{2EI_{BC}}{L_{BC}}(\theta_B + 2\theta_C) - \Omega_{CB} - M_C = 0 \qquad [14]$$

According to the slope-deflection method, Equations [11], [13], and [14] (the joint equilibrium equations) can be solved for the three joint rotations, as follows:

$$\theta_A = -\frac{L_{AB}[(3L_{AB}I_{BC} + 4L_{BC}I_{AB})(\Omega_{AB} + M_A) + L_{BC}I_{AB}(2\Omega_{BA} - 2\Omega_{BC} - \Omega_{CB}) - M_C L_{BC} I_{AB}]}{12EI_{AB}(L_{AB}I_{BC} + L_{BC}I_{AB})} \qquad [15]$$

$$\theta_B = -\frac{L_{AB}L_{BC}(\Omega_{AB} + 2\Omega_{BA} - 2\Omega_{BC} - \Omega_{CB} + M_A - M_C)}{6E(L_{AB}I_{BC} + L_{BC}I_{AB})} \qquad [16]$$

$$\theta_C = \frac{L_{BC}[(4L_{AB}I_{BC} + 3L_{BC}I_{AB})(\Omega_{CB} + M_C) + L_{AB}I_{BC}(2\Omega_{BC} - 2\Omega_{BA} - \Omega_{AB}) - M_A L_{AB}I_{BC}]}{12EI_{BC}(L_{AB}I_{BC} + L_{BC}I_{AB})} \qquad [17]$$

Substituting θ_A, θ_B and θ_C in the slope-deflection equations, we get the member-end moments as follows:

$$M_{AB} = -M_A \qquad [18]$$

$$M_{BA} = -\frac{I_{BC}L_{AB}(\Omega_{AB} + 2\Omega_{BA}) + I_{AB}L_{BC}(2\Omega_{BC} + \Omega_{CB}) + I_{BC}L_{AB}M_A + I_{AB}L_{BC}M_C}{2(I_{BC}L_{AB} + I_{AB}L_{BC})} \qquad [19]$$

$$M_{BC} = \frac{I_{BC}L_{AB}(\Omega_{AB} + 2\Omega_{BA}) + I_{AB}L_{BC}(2\Omega_{BC} + \Omega_{CB}) + I_{BC}L_{AB}M_A + I_{AB}L_{BC}M_C}{2(I_{BC}L_{AB} + I_{AB}L_{BC})} \qquad [20]$$

$$M_{CB} = M_C \qquad [21]$$

Of particular interest to us is Equation [19]. Since $M_B = M_{BA}$, we can write:

$$M_B = -\frac{I_{BC}L_{AB}(\Omega_{AB} + 2\Omega_{BA}) + I_{AB}L_{BC}(2\Omega_{BC} + \Omega_{CB}) + I_{BC}L_{AB}M_A + I_{AB}L_{BC}M_C}{2(I_{BC}L_{AB} + I_{AB}L_{BC})} \qquad [22]$$

Rearranging Equation [22], we get the three-moment equation in its general form.

$$\boxed{\frac{L_{AB}}{I_{AB}}M_A + 2M_B\left(\frac{L_{AB}}{I_{AB}} + \frac{L_{BC}}{I_{BC}}\right) + \frac{L_{BC}}{I_{BC}}M_C = -\frac{L_{AB}}{I_{AB}}(\Omega_{AB} + 2\Omega_{BA}) - \frac{L_{BC}}{I_{BC}}(2\Omega_{BC} + \Omega_{CB})} \qquad [23]$$

Note the three moments to the left of the equality sign. The equation defines a relationship among these moments, the beam lengths, the moments of inertia, and the loads.

Let us put this equation to use for the analysis of a continuous beam. Consider the two-span beam shown below, in which we have labeled the moments at A, B, and C as M_A, M_B, and M_C, respectively.

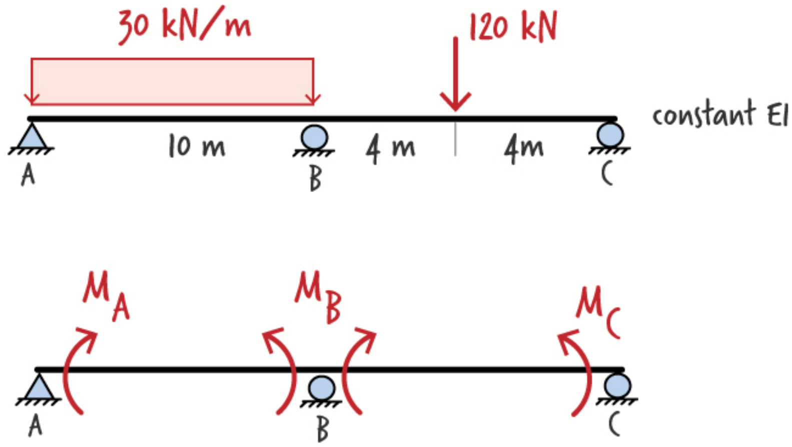


Figure 7: A two-span beam subjected to a distributed load and a concentrated load

Assuming the two beam segments have the same moment of inertia, Equation [24] simplifies to:

$$L_{AB}M_A + 2M_B(L_{AB} + L_{BC}) + L_{BC}M_C = -L_{AB}(\Omega_{AB} + 2\Omega_{BA}) - L_{BC}(2\Omega_{BC} + \Omega_{CB}) \quad [24]$$

Since the left segment of the beam is subjected to a uniformly distributed load, the fixed-end moments for the segment can be written as:

$$\Omega_{AB} = \Omega_{BA} = \frac{wL^2}{12} = \frac{(30)(10^2)}{12} = 250 \quad [25]$$

For the right segment of the beam, the fixed-end moments due to the concentrated load are:

$$\Omega_{BC} = \Omega_{CB} = \frac{PL}{8} = \frac{(120)(8)}{8} = 120 \quad [26]$$

Then, Equation [24] can be simplified to:

$$10M_A + 2M_B(10+8) + 8M_C = -10(250 + 2\times250) - 8(2\times120 + 120) \quad [27]$$

Or:

$$10M_A + 36M_B + 8M_C = -10380 \quad [28]$$

And since $M_A = M_C = 0$, we can write:

$$0+36M_B+0=-10380 \quad [29]$$

Solving the above equation for the unknown moment, we get: $M_B=-288.33$ kN.m. Now we can use the free-body diagram of beam segments, shown in Figure 8 to further analyze the beam.

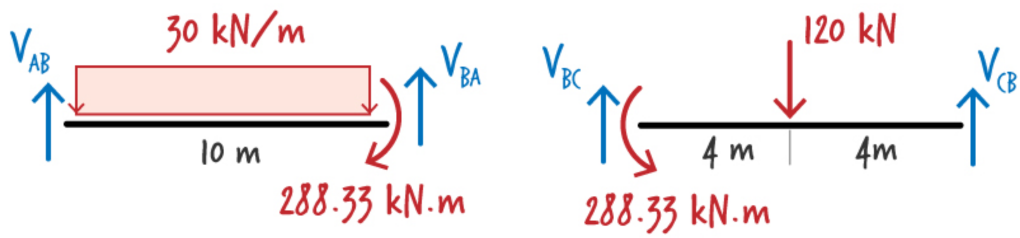


Figure 8: Free-body diagram of the beam segments in a two-span beam

The member-end shear forces in segment AB can be determined using the following static equilibrium equations.

$$10V_{BA}-288.33-30(10)(5)=0 \quad [30]$$

$$V_{AB}+V_{BA}-30(10)=0 \quad [31]$$

Or, $V_{AB}=121.17$ kN and $V_{BA}=178.83$ kN.

For segment BC, the equilibrium equations can be written as follows:

$$8V_{BC}-288.33-120(4)=0 \quad [32]$$

$$V_{BC}+V_{CB}-120=0 \quad [33]$$

Solving Equations [32] and [33] for the unknown shear forces, we get: $V_{BC}=96.04$ kN and $V_{CB}=23.96$ kN. As shown in Figure 9, the member-end shear forces yield the reaction forces.

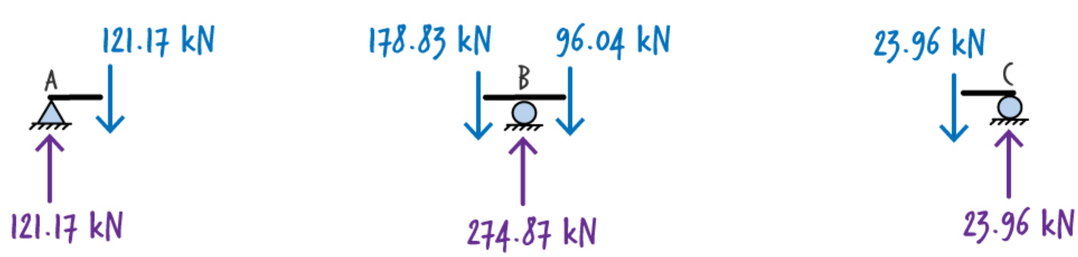


Figure 9: Shear forces acting at the joints of a two-span beam

The results of the analysis are depicted in Figure 10.

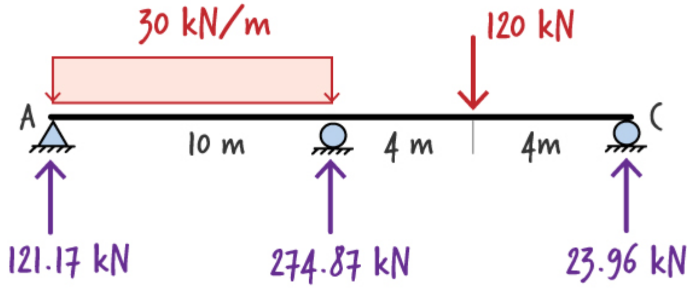


Figure 10: Calculated support reactions for a two-span beam

An alternative three-moment equation is worth mentioning. When simplified for the above example, Equation [23] can be written as:

$$10M_A + 36M_B + 8M_C = -10(\Omega_{AB} + 2\Omega_{BA}) - 8(2\Omega_{BC} + \Omega_{CB}) \qquad [34]$$

The alternative three-moment equation, which differs from Equation [34] only on the right of the equality sign, is as follows:

$$10M_A + 36M_B + 8M_C = -\frac{6A_{AB}\bar{x}_{AB}}{L_{AB}} - \frac{6A_{BC}\bar{x}_{BC}}{L_{BC}} \qquad [35]$$

Here, A_{AB} is the area under the moment diagram due to the distributed load and $\bar{x}_{AB}$ is the distance from the center of the moment diagram area to point A. Similarly, A_{BC} is the area under the moment diagram due to the load on segment BC and $\bar{x}_{BC}$ is the distance from the center of the moment diagram area to point C.

The relationship between the two versions of the three-moment equation can be established using algebraic manipulation. I'll leave that as an exercise problem for you.

Let's test Equation [35], the alternative three-moment equation. For the beam shown in Figure 7, the moment diagram for each beam segment is shown below. Note that here we are not drawing the moment diagram for the continuous beam. Rather, we are drawing the moment diagram for each beam segment, with the assumption that the segment is simply supported.

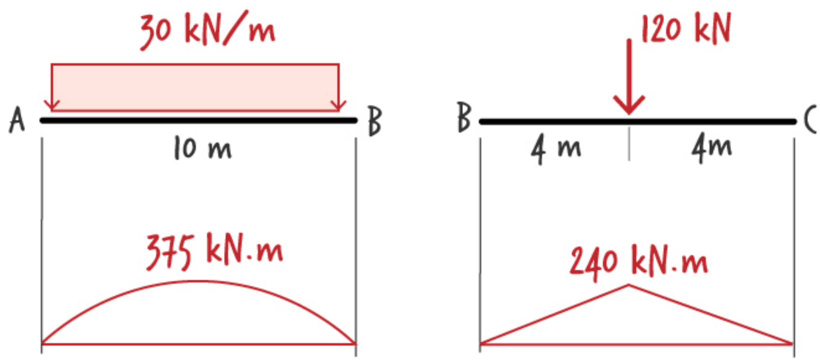


Figure 11: Bending moment diagrams for the isolated beam segments in a two-span beam

The area under the moment diagram for segment AB is: $(2/3)\times375\times10$, or 2500. The distance from the center of the area to point A is 5 meters.

For segment BC, the area under the moment diagram is 240(8/2) = 960, and the distance from the center of the area to point C is 4 meters. Therefore, the three-moment equation can be written as follows:

$$10M_A + 36M_B + 8M_C = -\frac{6(2500)(5)}{10} - \frac{6(960)(4)}{8} \qquad [36]$$

Or:

$$10M_A + 36M_B + 8M_C = -10380 \qquad [37]$$

Note that Equation [37] is the same as Equation [28], which was obtained using fixed-end moments. So, both formulations yield the same results.

The general form of this version of the three-moment equation is:

$$\boxed{\frac{L_{AB}}{I_{AB}}M_A + 2M_B\left(\frac{L_{AB}}{I_{AB}} + \frac{L_{BC}}{I_{BC}}\right) + \frac{L_{BC}}{I_{BC}}M_C = -\frac{6A_{AB}\bar{x}_{AB}}{I_{AB}L_{AB}} - \frac{6A_{BC}\bar{x}_{BC}}{I_{BC}L_{BC}}} \qquad [38]$$

We can use either Equation [23] or Equation [38] when analyzing beams using the three-moment method.

Let us consider another example. We wish to analyze the beam shown in Figure 12 using the three-moment method.

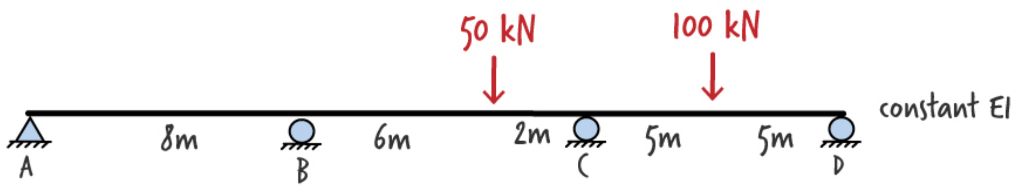


Figure 12: A three-span beam subjected to two concentrated loads

In this case, we need to write the three-moment equation twice, once for points A, B, and C, and a second time for points B, C, and D.

The simplified version of Equation [23], gives us:

$$8M_A + 2M_B(8+8) + 8M_C = -8(\Omega_{AB} + 2\Omega_{BA}) - 8(2\Omega_{BC} + \Omega_{CB}) \qquad [39]$$

Since segment AB is not subjected to any loads, it follows that $\Omega_{AB} = \Omega_{BA} = 0$. The fixed-end moments for segment BC are given in Figure 13, in which $\Omega_{BC} = 18.75$ and $\Omega_{BC} = 56.25$.

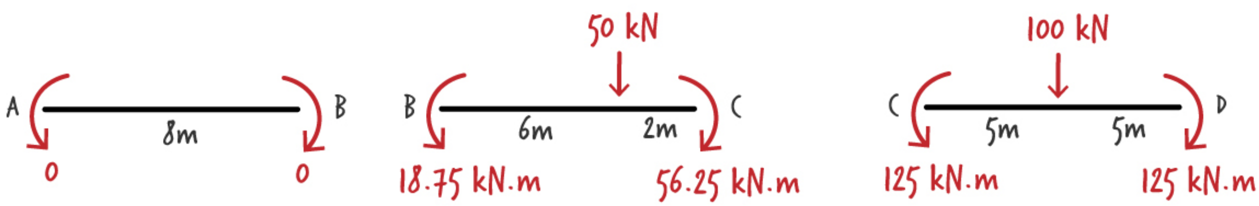


Figure 13: Fixed-end moments for the beam segments in a three-span beam

Therefore, Equation [39] becomes:

$$8M_A + 2M_B(8+8) + 8M_C = -8(0+0) - 8(2\times 18.75 + 56.25) \qquad [40]$$

Or:

$$8M_A + 32M_B + 8M_C = -750 \qquad [41]$$

Now let's see if the alternative formulation gives us the same equation. To use Equation [38], we need to draw the moment diagram for each beam segment, assuming that the segment is a simply supported beam. The diagrams are shown below.

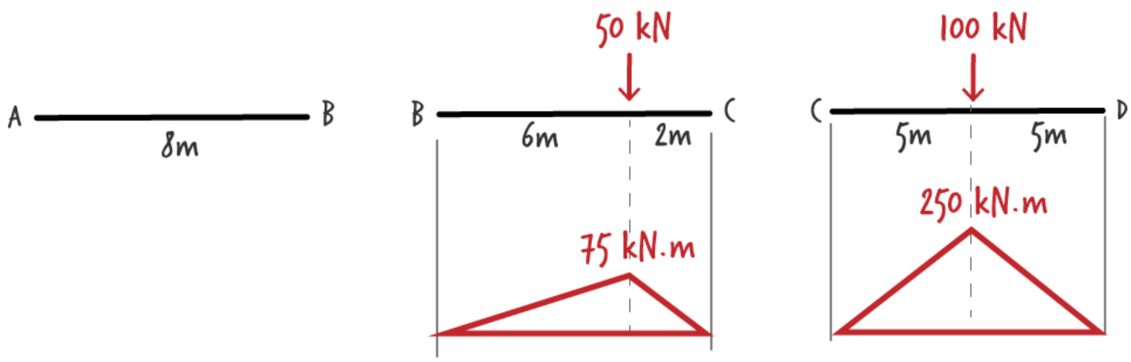


Figure 14: Bending moment diagrams for the isolated beam segments in a three-span beam

Assuming a constant moment of inertia for the entire beam, Equation [38] becomes:

$$L_{AB}M_A + 2M_B(L_{AB} + L_{BC}) + L_{BC}M_C = -\frac{6A_{AB}\bar{x}_{AB}}{L_{AB}} - \frac{6A_{BC}\bar{x}_{BC}}{L_{BC}} \qquad [42]$$

Since there is no moment diagram in segment AB, the equation simplifies to:

$$L_{AB}M_A + 2M_B(L_{AB} + L_{BC}) + L_{BC}M_C = 0 - \frac{6A_{BC}\bar{x}_{BC}}{L_{BC}} \qquad [43]$$

Quantity $A_{BC}\bar{x}_{BC}$ in the above equation, which represents the area under the moment diagram in segment BC, times the distance from the center of the area to point C, can be determined as follows:

$$(75)(\frac{6}{2})(2+2) + (75)(\frac{2}{2})(\frac{4}{3}) = 1000 \qquad [44]$$

Hence, Equation [43] becomes:

$$8M_A + 2(8+8)M_B + 8M_C = -\frac{6 \times 1000}{8} \qquad [45]$$

Or:

$$8M_A + 32M_B + 8M_C = -750 \qquad [46]$$

As you see, Equation [46] is identical to Equation [41]. That is, we can use either formulation to write the three-moment equation.

The three-moment equation for points B, C, and D can be written as:

$$L_{BC}M_B + 2M_C(L_{BC} + L_{CD}) + L_{CD}M_D = -L_{BC}(\Omega_{BC} + 2\Omega_{CB}) - L_{CD}(2\Omega_{CD} + \Omega_{DC}) \qquad [47]$$

Alternatively, we can write the equation as:

$$L_{BC}M_B + 2M_C(L_{BC} + L_{CB}) + L_{CD}M_D = -\frac{6A_{BC}\overline{x}_{BC}}{L_{BC}} - \frac{6A_{CD}\overline{x}_{CD}}{L_{CD}} \qquad [48]$$

Knowing the fixed-end moments for the beam segments (see Figure 13), Equation [47] becomes:

$$8M_B + 2M_C(8+10) + 10M_D = -8(18.75 + 2\times 56.25) - 10(2\times 125 + 125) \qquad [49]$$

Or:

$$8M_B + 36M_C + 10M_D = -4800 \qquad [50]$$

Using the moment diagrams shown in Figure 14, the alternative formulation (Equation [48]), can be written as follows:

$$8M_B + 2M_C(8+10) + 10M_D = -6(\frac{75(6/2)(4) + 75(2/2)(6+2/3)}{8}) - 6(\frac{250(10/2)(5)}{10}) \qquad [51]$$

Or:

$$8M_B + 36M_C + 10M_D = -4800 \qquad [52]$$

Again, note that the above equation is identical to the equation obtained using the fixed-end moment values.

Since $M_A = M_D = 0$, our three-moment equations for the beam become:

$$0 + 32M_B + 8M_C = -750 \qquad [53]$$

$$8M_B + 36M_C + 0 = -4800 \qquad [54]$$

Solving for the unknown moments, we get: $M_B = 10.48 \text{ kN.m}$ and $M_C = -135.66 \text{ kN.m}$.

Now we can determine the beam's support reactions using the static equilibrium equations, as depicted below.

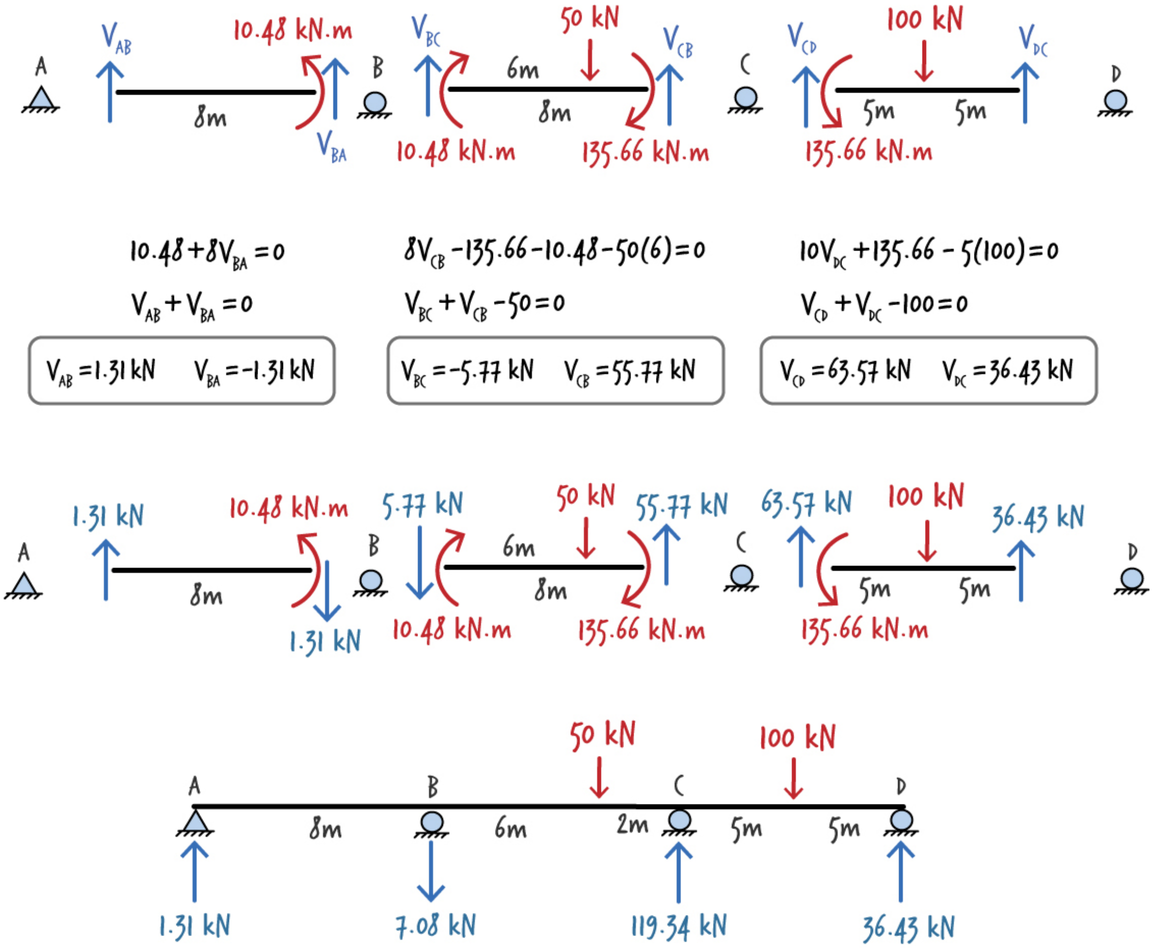
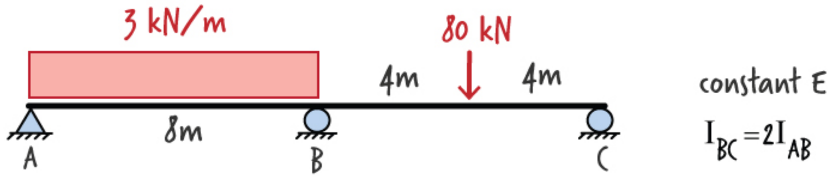


Figure 15: The analysis details for a three-span beam

We will expand our discussion on the use of the three-moment equation in the next lecture.

Exercise Problems: Analyze the following beams using the three-moment equation method.

1)



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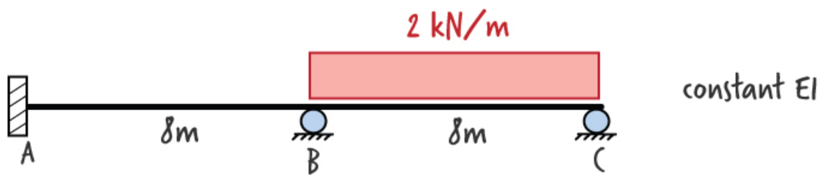

3)