Structural Analysis

Lecture Series

SA61: Three-Moment Equation: Part II

Analysis of Continuous Beams with Support Settlement

This document is a written version of video lecture SA61, which can be found online at the web addresses listed below.

Educative Technologies, LLC

http://www.Lab101.Space
https://www.youtube.com/c/drstructure

2019

Structural Analysis – SA61

The three-moment equation for the analysis of continuous beams with support settlement

Prerequisite: Lecture SA60

The three-moment equation is a single algebraic expression that establishes a relationship among the moment values at three consecutive points in a beam. Lecture SA60 covered the use of the three-moment equation for analyzing beams with no support settlements. This lecture expands the previous discussion by introducing support settlements into the equation.

Consider the continuous two-span beam shown below.

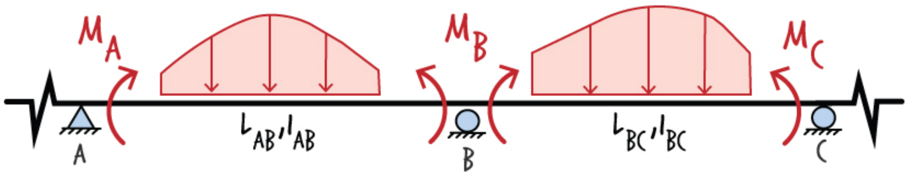

Figure 1: A two-span beam

Suppose segments AB and BC are subjected to a counterclockwise rotation caused by support displacements at B and C, as depicted in Figure 2.

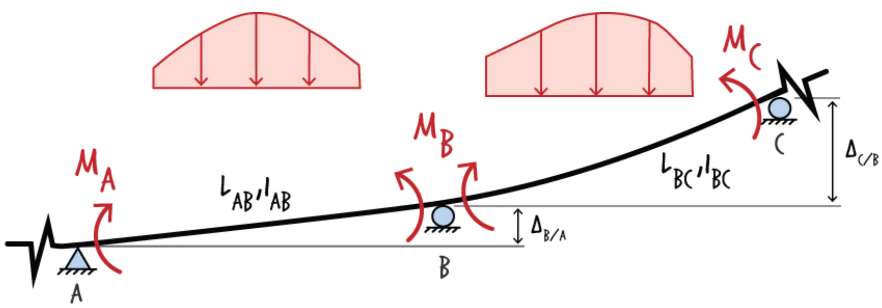

Figure 2: A two-span beam with support settlements

The differential vertical displacement in segment AB, denoted by $\Delta_{B/A}$, is the difference between the settlements at A and B. Similarly, $\Delta_{C/B}$ denotes the differential vertical displacement for segment BC. These differential displacements, which represent the support settlements, can be introduced in the three-moment equation via the slope-deflection formulation.

The generalized form of the slope-deflection equations for segment AB are given below.

$$M_{AB} = \frac{2EI_{AB}}{L_{AB}}(2\theta_A + \theta_B - \frac{3\Delta_{B/A}}{L_{AB}}) + \Omega_{AB} \quad [1]$$

$$M_{BA} = \frac{2EI_{AB}}{L_{AB}}(\theta_A + 2\theta_B - \frac{3\Delta_{B/A}}{L_{AB}}) - \Omega_{BA} \quad [2]$$

In the above equations, Ω symbolizes fixed-end moments.

For member BC, the slope-deflection equations can be written as follows.

$$M_{BC} = \frac{2EI_{BC}}{L_{BC}}(2\theta_B + \theta_C - \frac{3\Delta_{C/B}}{L_{BC}}) + \Omega_{BC} \quad [3]$$

$$M_{CB} = \frac{2EI_{BC}}{L_{BC}}(\theta_B + 2\theta_C - \frac{3\Delta_{C/B}}{L_{BC}}) - \Omega_{CB} \quad [4]$$

Figure 3 shows two different representations of the member-end moments. In Figure 3a, the moments are shown using the three-moment equation sign convention. Figure 3b shows the moments using the slope-deflection sign convention.

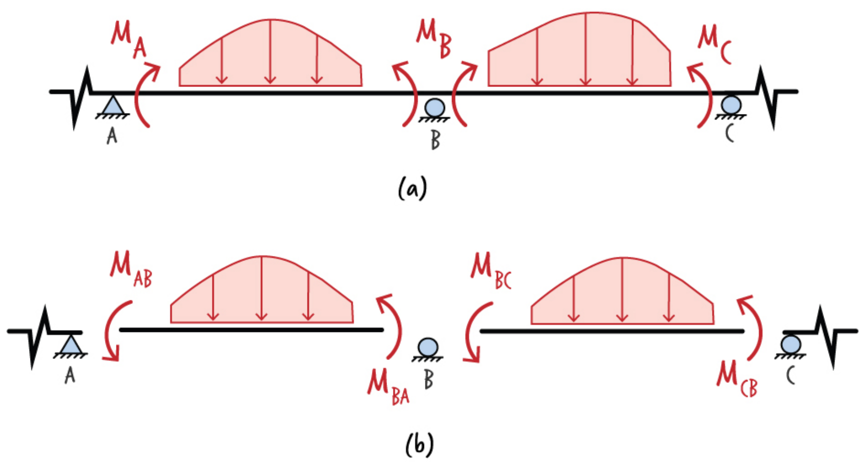

Figure 3: Moment representations for the three-moment and slope-deflection formulations

Comparing the two sets of moments, we can see that: $M_{AB} = -M_A$, $M_{BA} = M_B$, $M_{BC} = -M_B$, and $M_{BC} = M_C$. Therefore, the following joint equilibrium equations hold true.

$$M_{AB} = -M_A \quad [5]$$

$$M_{BA} + M_{BC} = 0 \quad [6]$$

$$M_{CB} = M_C \quad [7]$$

By substituting the slope-deflection equations in Equations [5], [6], and [7], the following equations result:

$$M_A + \frac{2EI_{AB}}{L_{AB}}(2\theta_A + \theta_B - \frac{3\Delta_{B/A}}{L_{AB}}) + \Omega_{AB} = 0 \quad [8]$$

$$\frac{2EI_{AB}}{L_{AB}}(\theta_A + 2\theta_B - \frac{3\Delta_{B/A}}{L_{AB}}) + \frac{2EI_{BC}}{L_{BC}}(\theta_C + 2\theta_B - \frac{3\Delta_{C/B}}{L_{BC}}) + \Omega_{BC} - \Omega_{BA} = 0 \quad [9]$$

$$-M_C+\frac{2EI_{BC}}{L_{BC}}(2\theta_C+\theta_B-\frac{3\Delta_{C/B}}{L_{BC}})-\Omega_{CB}=0 \qquad [10]$$

We can determine the slopes at A, B, and C (θ_A, θ_B, and θ_C) by solving Equations [8], [9], and [10] simultaneously. The resulting algebraic expressions for the three slopes are given below.

$$\theta_A=\frac{3I_{BC}\Delta_{B/A}}{2(I_{BC}L_{AB}+I_{AB}L_{BC})}-\frac{I_{BC}L_{AB}^2(\Omega_{AB}+M_A)}{4EI_{AB}(I_{BC}L_{AB}+I_{AB}L_{BC})}-\frac{I_{BC}L_{AB}\Delta_{C/B}}{2L_{BC}(I_{BC}L_{AB}+I_{AB}L_{BC})}-\frac{I_{AB}L_{BC}\Delta_{B/A}}{L_{AB}(I_{BC}L_{AB}+I_{AB}L_{BC})}-\frac{L_{AB}L_{BC}(4\Omega_{AB}+2\Omega_{BA}-2\Omega_{BC}-\Omega_{CB}+4M_A-M_C)}{12E(I_{BC}L_{AB}+I_{AB}L_{BC})} \qquad [8]$$

$$\theta_B=\frac{(I_{BC}L_{AB}^2\Delta_{C/B}+I_{AB}L_{BC}^2\Delta_{B/A})}{L_{BC}L_{AB}(I_{BC}L_{AB}+I_{AB}L_{BC})}+\frac{L_{AB}L_{BC}(\Omega_{AB}-\Omega_{CB}+\Omega_{BA}-\Omega_{BC}+M_A-M_C)}{6E(I_{BC}L_{AB}+I_{AB}L_{BC})} \qquad [9]$$

$$\theta_C=\frac{3I_{AB}\Delta_{C/B}}{2(I_{BC}L_{AB}+I_{AB}L_{BC})}+\frac{I_{BC}L_{AB}\Delta_{C/B}}{L_{BC}(I_{BC}L_{AB}+I_{AB}L_{BC})}-\frac{I_{AB}L_{BC}\Delta_{B/A}}{2L_{AB}(I_{BC}L_{AB}+I_{AB}L_{BC})}-\frac{L_{AB}L_{BC}(\Omega_{AB}+2\Omega_{BA}+2\Omega_{BC}+4\Omega_{CB}+4M_C-M_A)}{12E(I_{BC}L_{AB}+I_{AB}L_{BC})}+\frac{I_{AB}L_{BC}^2(\Omega_{CB}+M_C)}{4EI_{BC}(I_{BC}L_{AB}+I_{AB}L_{BC})} \qquad [10]$$

Substituting Equations [8] through [10] in Equation [2] results in Equation [11].

$$M_{BA}=\frac{6EI_{AB}I_{BC}(\Delta_{C/B}L_{AB}-\Delta_{B/A}L_{BC})}{2L_{AB}L_{BC}(I_{BC}L_{AB}+I_{AB}L_{BC})}-\frac{L_{AB}I_{BC}(\Omega_{AB}+2\Omega_{BA}+M_A)+L_{BC}I_{AB}(2\Omega_{BC}+\Omega_{CB}+M_C)}{2(I_{BC}L_{AB}+I_{AB}L_{BC})} \qquad [11]$$

And since $M_{BA}=M_B$ (see Figure 3), Equation [11] can be rewritten in the following form.

$$M_B=\frac{6EI_{AB}I_{BC}(\Delta_{C/B}L_{AB}-\Delta_{B/A}L_{BC})}{2L_{AB}L_{BC}(I_{BC}L_{AB}+I_{AB}L_{BC})}-\frac{L_{AB}I_{BC}(\Omega_{AB}+2\Omega_{BA}+M_A)+L_{BC}I_{AB}(2\Omega_{BC}+\Omega_{CB}+M_C)}{2(I_{BC}L_{AB}+I_{AB}L_{BC})} \qquad [12]$$

By rearranging the terms of the above equation, we can obtain the generalized form of the three-moment equation, as shown in Equation [13].

$$\frac{L_{AB}}{I_{AB}}M_A+2(\frac{L_{AB}}{I_{AB}}+\frac{L_{BC}}{I_{BC}})M_B+\frac{L_{BC}}{I_{BC}}M_C=\frac{6E\Delta_{C/B}}{L_{BC}}-\frac{6E\Delta_{B/A}}{L_{AB}}-\frac{L_{AB}}{I_{AB}}(\Omega_{AB}+2\Omega_{BA})-\frac{L_{BC}}{I_{BC}}(2\Omega_{BC}+\Omega_{CB}) \quad [13]$$

Let's see how Equation [13] can be used to analyze continuous beams with support reactions.

Example 1: Consider the beam shown below. Suppose the support at B has settled 10 mm. We wish to calculate the support reactions due to the settlement without considering the applied loads. The beam has a constant EI in which E is 200 GPa, and $I = 0.0001\ m^4$.

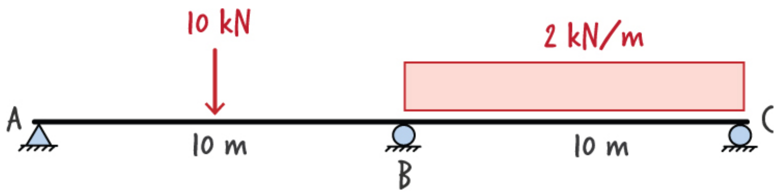

Figure 4: A two-span continuous beam

We start by drawing the internal moments at the supports, as shown below.

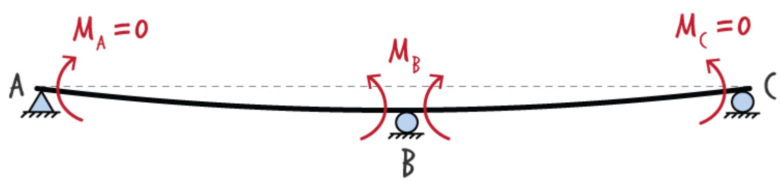

Figure 5: Internal bending moments in a two-span beam

In this case, since there is a pin at A and a roller at C, $M_A=M_C=0$. Furthermore, since we are not considering the effect of the applied loads on the support reaction forces, all the fixed-end moment terms in Equation [13] are assumed to be zero. Therefore, the three-moment equation can be written in the simplified form shown below.

$$2(\frac{L_{AB}}{I_{AB}}+\frac{L_{BC}}{I_{BC}})M_B=\frac{6E\Delta_{C/B}}{L_{BC}}-\frac{6E\Delta_{B/A}}{L_{AB}} \quad [14]$$

To determine $\Delta_{A/B}$ and $\Delta_{C/B}$, we need to examine each beam segment separately. Segment AB rotates clockwise as a result of the settlement at support B (see Figure 6). And since clockwise is considered negative direction for rotation according to the slope-deflection sign convention, $\Delta_{B/A}$ is negative. That is, $\Delta_{B/A}=-10\text{ mm}=-0.01\text{ m}$.

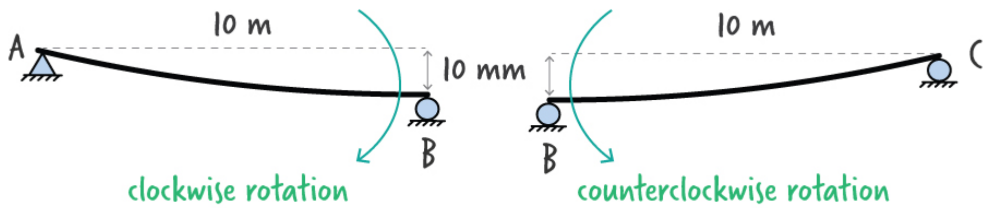

Figure 6: Segment rotations in a two-span beam due to a support settlement

And since segment BC rotates counterclockwise due to the settlement at B, $\Delta_{C/B}$ is considered positive. Hence, $\Delta_{C/B} = 0.01\text{ m}$. Consequently, Equation [14] can be expanded as follows.

$$2\left(\frac{10}{0.0001}+\frac{10}{0.0001}\right)M_B = \frac{6(200\times10^9)(0.01)}{10} - \frac{6(200\times10^9)(-0.01)}{10} \qquad [15]$$

Solving Equation [15] for M_B, we get: $M_B = 6000\text{ N.m}$, or $M_B = 6\text{ kN.m}$.

Figure 7 shows the resulting free-body diagrams for Segments AB and BC.

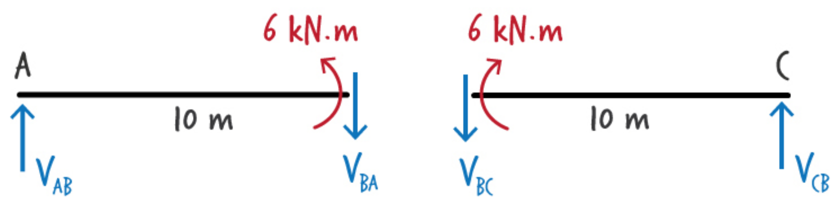

Figure 7: Internal bending moments in a two-span beam

The above diagrams can be used to calculate the member-end shear forces. This is done by writing and solving the static equilibrium equations for each segment. For segment AB, the equations can be written in the following manner.

$$6 - 10V_{BA} = 0 \qquad [16]$$

$$V_{AB} - V_{BA} = 0 \qquad [17]$$

Solving the above equations for the two shear forces, we get: $V_{BA} = 0.6\text{ kN}$ and $V_{AB} = 0.6\text{ kN}$. Similarly, the equilibrium equations for segment BC are as follows.

$$6 - 10V_{BC} = 0 \qquad [18]$$

$$V_{CB} - V_{BC} = 0 \qquad [19]$$

Equations [18] and [19] yield: $V_{BC} = 0.6\ \text{kN}$ and $V_{CB} = 0.6\ \text{kN}$.

The reaction force at A equals V_{AB} . At B, the reaction force is the sum of V_{BA} and V_{BC} . And at C, the reaction equals V_{CB} . The following diagram shows the support reactions due to the settlement at B.

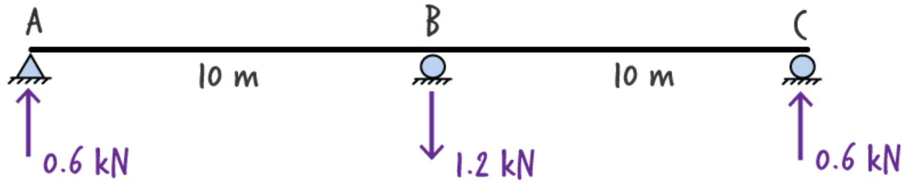

Figure 8: Support reaction due to a support settlement in a two-span beam

Example 2: Consider the beam shown below. It rests on a fixed support at A and on rollers at B and C. Suppose the support at B has a downward settlement of 10 mm, and the support at C has settled 4 mm. We wish to calculate the support reactions due to the applied load and the settlements. The beam has a constant EI in which E is 200 GPa and $I = 0.0001\ m^4$.

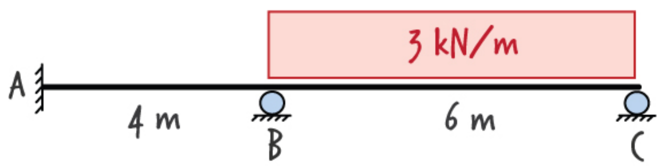

Figure 9: A continuous beam with a fixed support

To use the three-moment equation, we need to replace the fixed support at A with a fictitious infinitely rigid beam segment (see Figure 10).

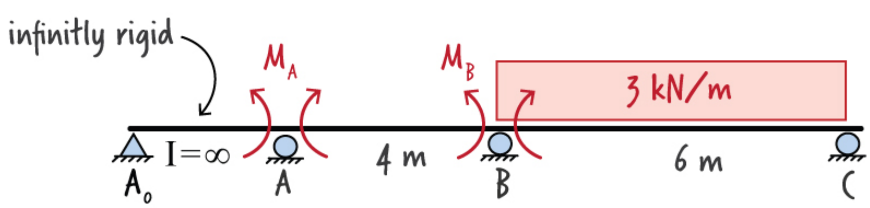

Figure 9: A continuous beam with a fictitious infinitely rigid beam segment

Now we can write two three-moment equations: one equation for the left two segments and another equation for the right two segments.

For the left two segments, Equation [13] can be written as shown below.

$$\frac{L_{A_0A}}{I_{A_0A}}M_{A_0}+2\left(\frac{L_{A_0A}}{I_{A_0A}}+\frac{L_{AB}}{I_{AB}}\right)M_A+\frac{L_{AB}}{I_{AB}}M_B=\frac{6E\Delta_{B/A}}{L_{AB}}-\frac{6E\Delta_{A/A_0}}{L_{A_0A}}-\frac{L_{A_0A}}{I_{A_0A}}(\Omega_{A_0A}+2\Omega_{AA_0})-\frac{L_{AB}}{I_{AB}}(2\Omega_{AB}+\Omega_{BA}) \quad [20]$$

Since $M_{A_0}=0$, $I_{A_0A}=\infty$, and fixed-end moments Ω_{A_0A}, Ω_{AA_0}, Ω_{AB} and Ω_{BA} are zero, the above equation simplifies to Equation [21].

$$2\left(\frac{L_{AB}}{I_{AB}}\right)M_A+\frac{L_{AB}}{I_{AB}}M_B=\frac{6E\Delta_{B/A}}{L_{AB}}-\frac{6E\Delta_{A/A_0}}{L_{A_0A}} \quad [21]$$

Assuming that δ_{A_0}, δ_A, and δ_B are vertical settlements at supports A_0, A, and B respectively, we can determine Δ_{A/A_0} and $\Delta_{B/A}$ using the following equations.

$$\Delta_{A/A_0}=\delta_A-\delta_{A_0}=0-0=0 \quad [22]$$

$$\Delta_{B/A}=\delta_A-\delta_B=(-0.01)-0=-0.01 \quad [23]$$

By substituting the values for Δ_{A/A_0}, $\Delta_{B/A}$, I_{AB}, L_{AB}, and E in Equation [21], the following equation results.

$$2\left(\frac{4}{0.0001}\right)M_A+\frac{4}{0.0001}M_B=\frac{6(200\times10^6)(-0.01)}{4} \quad [24]$$

In its simplified form, Equation [24] turns into Equation [25], which is one of the two equations we need to solve the problem.

$$2M_A+M_B=-75 \quad [25]$$

The three-moment equation for the two right segments is shown below.

$$\frac{L_{AB}}{I_{AB}}M_A+2\left(\frac{L_{AB}}{I_{AB}}+\frac{L_{BC}}{I_{BC}}\right)M_B+\frac{L_{BC}}{I_{BC}}M_C=\frac{6E\Delta_{C/B}}{L_{BC}}-\frac{6E\Delta_{B/A}}{L_{AB}}-\frac{L_{AB}}{I_{AB}}(\Omega_{AB}+2\Omega_{BA})-\frac{L_{BC}}{I_{BC}}(2\Omega_{BC}+\Omega_{CB}) \quad [26]$$

Since segment BC is subjected to a distributed load, the fixed-end moments for the segment are not zero; instead, they are: $\Omega_{BC}=\Omega_{CB}=(3)(6^2)/12=9$ kN.m. However, the fixed-end moments for segment AB are zero, as the segment is not subjected to any loads. Therefore, we can write: $\Omega_{AB}=\Omega_{BA}=0$.

Furthermore, $\Delta_{B/A}$ and $\Delta_{C/B}$ can be calculated as shown below.

$$\Delta_{B/A} = \delta_B - \delta_A = (-0.01) - 0 = -0.01 \quad [27]$$
$$\Delta_{C/B} = \delta_C - \delta_B = (-0.004) - (-0.01) = 0.006 \quad [28]$$

After making the necessary substitutions in Equation [26], the following equation results.

$$\frac{4}{0.0001}M_A + 2\left(\frac{4}{0.0001} + \frac{6}{0.0001}\right)M_B = 6(200\times10^6)\left(\frac{0.006}{6} + \frac{0.01}{4}\right) - \frac{6}{0.0001}(2\times9+9) \quad [29]$$

Which can be simplified to:

$$M_A + 5M_B = 64.5 \quad [30]$$

By solving Equations [25] and [30] simultaneously, we can determine M_A and M_B as follows: $M_A = -44.83$ kN.m and $M_B = 22.67$ kN.m. Now the resulting free-body diagrams for the two beam segments can be drawn as shown below.

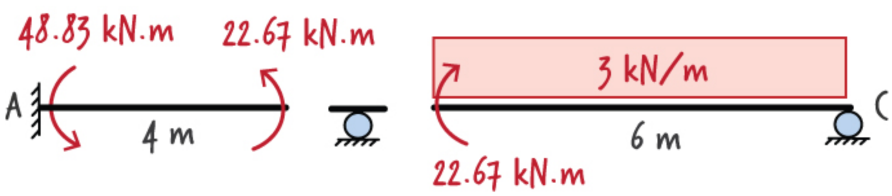

Figure 10: Internal bending moments in a continuous beam with a fixed end

Using the static equilibrium equations, we can determine the member-end shear forces and the support reactions for the beam. Figure 11 shows the resulting reaction forces for the beam.

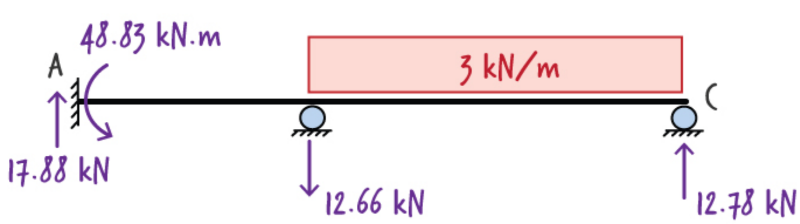

Figure 11: Support reactions for a continuous beam with a fixed end