Structural Analysis

Lecture Series

SA62: Cables

Introduction

This document is a written version of video lecture SA62, which can be found online at the web addresses listed below.

Educative Technologies, LLC

http://www.Lab101.Space

https://www.youtube.com/c/drstructure

2019

Structural Analysis – SA62
Cables – Introduction

The Golden Gate Bridge is an iconic structure that utilizes cables to carry loads (see Figure 1). Such cables are routinely used in bridges and other types of structures for load transfer. This lecture series provides an introduction to the analysis of cables.

Figure 1: The Golden Gate Bridge

As shown in the figure above, the Golden Gate Bridge is a suspension bridge. Its deck is suspended from a pair of main cables using a series of vertical hangers. This arrangement allows most of the bridge load to be transferred to the main cables, which in turn transfer the load to the towers located at the ends of the bridge. In a scenario like this, given the close proximity of the hangers, we can assume that the main cables are subjected to a distributed load. Furthermore, since the weight of each cable is insignificant compared to the load it must carry, we can neglect the cable's own weight when we analyze the system. A line drawing illustrating one of the bridge's main cables is shown in Figure 2.

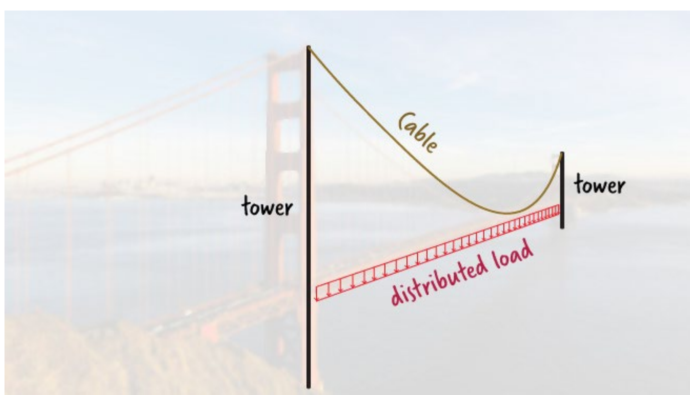


Figure 2: A line drawing characterization of a main cable in a suspension bridge

Let's now consider the pedestrian bridge shown in Figure 3. Note how the surface of the bridge deck follows the geometric shape of the cable. In this case, we can conceptualize the entire system as a cable hanging freely from its ends, carrying its own weight.

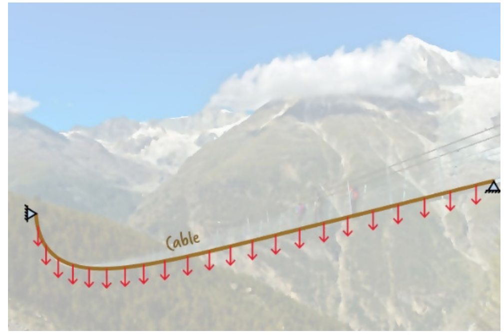


Figure 3: Charles Kuonen pedestrian bridge and its line drawing characterization

The behavior of the cable shown above is similar to the behavior of a power line hanging from two supporting transmission towers (see Figure 4).

Figure 4: A power line and its line drawing characterization as a cable hanging freely under its own weight

Before we go further, we should make a distinction between a weightless cable supporting a linear load and a cable carrying only its own weight. The two cables we just described differ in their shapes: when a cable carries a linear load distributed along the horizontal axis (e.g. the main cables in the Golden Gate Bridge), its shape can be defined using a parabola. On the other hand, when a cable hangs freely under its own weight (e.g. the Charles Kuonen pedestrian bridge), it takes the shape of a catenary.

Let's examine this difference more closely. Consider the cable shown in Figure 5. It is suspended from its two ends, creating a configuration resembling that of the Golden Gate Bridge. For this cable, we are assuming that the bridge deck is exerting a uniformly distributed load of **w** on the cable along the x-axis.

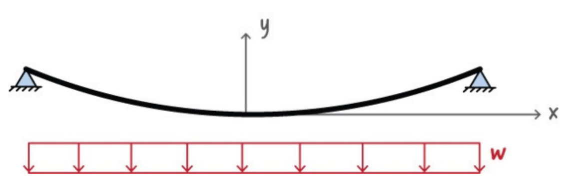


Figure 5: A cable subjected to a uniformly distributed load

If we place the origin of the coordinate system at the lowest point of the cable, we can draw the free-body diagram of the segment just to the right of the origin. This free-body diagram is shown in Figure 6.

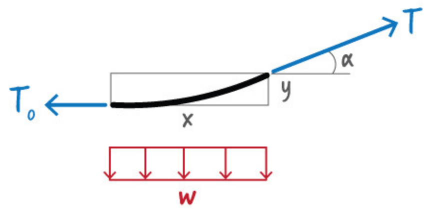


Figure 6: The free-body diagram of a segment of a cable subjected to a uniformly distributed load

In the above diagram, T_0 is the tension force in the cable at its lowest point, T is the tension force at the right end of the segment, and α denotes the angle that the cable makes with the horizontal axis at its right end.

Since the segment has to remain in equilibrium, the sum of the forces in the horizontal and vertical directions must be zero. Therefore, we can write two equilibrium equations, as shown below.

$$\sum F_x = 0 \quad \Rightarrow T\cos\alpha = T_0 \qquad [1]$$

$$\sum F_y = 0 \quad \Rightarrow T\sin\alpha = w\,x \qquad [2]$$

Dividing Equation [2] by Equation [1], we get the following:

$$\frac{T\sin\alpha}{T\cos\alpha} = \frac{w\,x}{T_0} \quad \Rightarrow \tan\alpha = \frac{w}{T_0}x \qquad [3]$$

Since $\tan\alpha$ can be expressed as the change in y with respect to the change in x (i.e., dy/dx), Equation [3] can be rewritten as follows:

$$\frac{dy}{dx} = \frac{w}{T_0}x \qquad [4]$$

This first order differential equation can be easily solved for y. Let's rewrite Equation [4] as shown below.

$$dy = \frac{w}{T_0}x\,dx \qquad [5]$$

Applying the integral operator to both sides of Equation [5], we get the following:

$$\int dy = \int \frac{w}{T_0}x\,dx \quad \Rightarrow y = \frac{w}{2T_0}x^2 + C \qquad [6]$$

The integration constant (C) in Equation [6] can be determined using a boundary condition. In this case, we know that at the origin, where $x = 0$, y is also zero. Substituting zero for x and y in the above equation, we can determine C.

$$y = \frac{w}{2T_0} x^2 + C \quad \Rightarrow 0 = \frac{w}{2T_0}(0)^2 + C \quad \Rightarrow C = 0 \qquad [7]$$

Therefore, Equation [6] can be written as follows:

$$y = \frac{w}{2T_0} x^2 \qquad [8]$$

Equation [8] is a parabolic function that describes the shape of the cable shown in Figure 5.

Now let's examine the case in which the cable hangs freely under its own weight. Consider the cable shown in Figure 7.

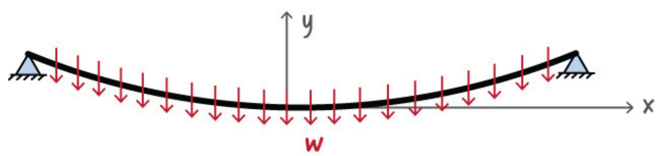


Figure 7: A cable hanging freely from its ends and subjected to its own weight

Note that the weight of the cable is not distributed along the x-axis but rather along the arc length of the cable. Let’s place our origin at the lowest point of the cable just like we did before. The free-body diagram for the segment of the cable to the right of the origin is shown in Figure 8.

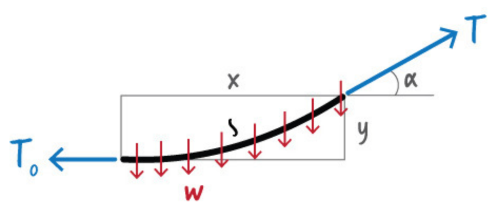


Figure 8: The free-body diagram of a cable subjected to its own weight

Note that the arc length of the segment in the figure above is denoted by s. We can write two equilibrium equations for the free-body diagram shown in Figure 8 as follows.

$$\sum F_x = 0 \quad \Rightarrow T\cos\alpha = T_0 \qquad [9]$$

$$\sum F_y = 0 \quad \Rightarrow T\sin\alpha = w\,s \qquad [10]$$

Dividing Equation [10] by Equation [9], we obtain Equation [11].

$$\frac{T\sin\alpha}{T\cos\alpha} = \frac{w\,s}{T_0} \quad \Rightarrow \tan\alpha = \frac{w}{T_0}s \quad \Rightarrow \frac{dy}{dx} = \frac{w}{T_0}s \qquad [11]$$

In order to solve the above differential equation, we need to replace s with x and y. We can do this using the Pythagorean theorem. Note that ds can be viewed as the hypotenuse of a right triangle with height dy and base dx. Therefore, we can write the following:

$$ds^2 = dx^2 + dy^2 \quad \Rightarrow ds = \sqrt{dx^2 + dy^2} \qquad [12]$$

If we take the derivative of both sides of Equation [11] with respect to *x*, we get the following equation.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{w}{T_0}s\right) \quad \Rightarrow \frac{d^2y}{d^2x} = \frac{w}{T_0}\frac{ds}{dx} \qquad [13]$$

Substituting Equation [12] into Equation [13], we get the following second-order differential equation.

$$\frac{d^2y}{d^2x} = \frac{w}{T_0}\frac{\sqrt{dx^2 + dy^2}}{dx} = \frac{w}{T_0}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad [14]$$

When solved, the above equation yields the following solution.

$$y = \frac{T_0}{w}\cosh\left(\frac{w}{T_0}x\right) - \frac{T_0}{w} \qquad [15]$$

The above equation is that of a catenary. It is used to describe the shape of a cable hanging freely under its own weight.

To summarize, when analyzing cable systems subjected to distributed loads, depending on the source and nature of the loads, we may have to use a different mathematical function to describe the shape of each cable. This lecture series deals with the analysis of such cable systems.

Let's start with a simple example. Consider a weightless cable spanning a distance of 10 meters between two poles. Attached to the cable are two traffic lights. The traffic lights cause the cable to settle, forming three straight segments (see Figure 9). Each traffic light weighs 400 Newtons.

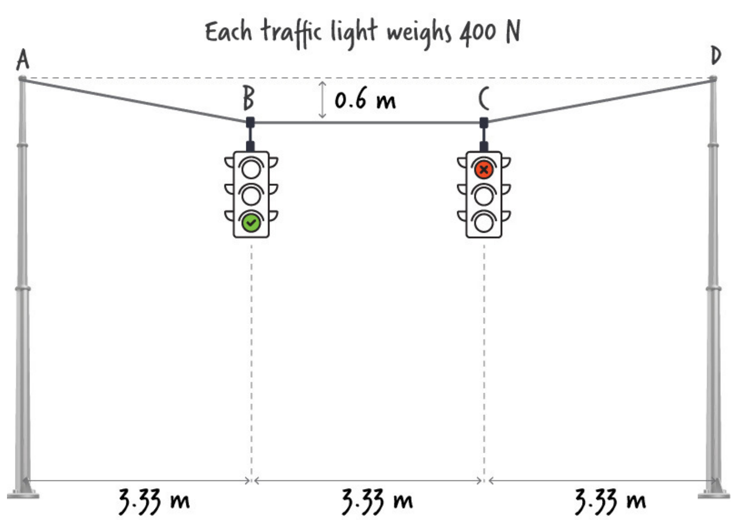


Figure 9: Two equidistant traffic lights hanging from a cable between two poles

The two lights divide the span of the road into three equal distances. Due to the symmetrical placement of the lights and the position of the cable, we know that points B and C displace downward the same amount. When measured, the vertical distance from the top of each pole to points B and C is 0.6 meter.

We want to determine the tension force in each segment of the cable.

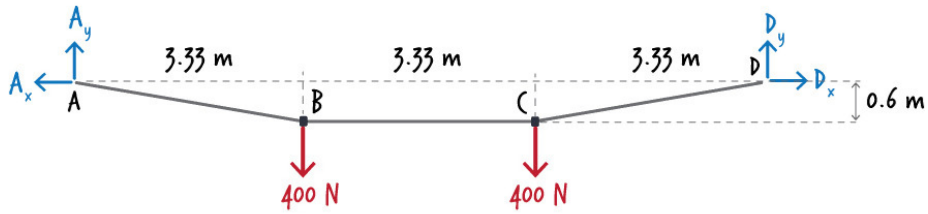


Figure 10: Free-body diagram for a cable system under symmetrical loading

The solution for this problem is rather straightforward. Let's start by drawing the free-body diagram of the cable. As depicted in Figure 10, there are two support reactions at each end of the cable. Therefore, the three static equilibrium equations can be written as follows:

$$\sum F_x = 0 \quad \Rightarrow D_x - A_x = 0 \tag{16}$$

$$\sum F_y = 0 \quad \Rightarrow A_y + D_y - 400 - 400 = 0 \tag{17}$$

$$\sum M @ A = 0 \quad \Rightarrow 10D_y - 400(3.33) - 400(6.66) = 0 \tag{18}$$

Using Equations [17] and [18], we can solve for the vertical support reactions (i.e. A_y and D_y). However, the horizontal reactions (i.e. A_x and D_x) cannot be determined from Equation [16] since it has too many unknowns.

For this problem, however, we can determine the unknown forces without resorting to Equations [16] through [18]. Let's cut the cable in segment BC and draw the free-body diagram for the left side of the cable system as shown in Figure 11.

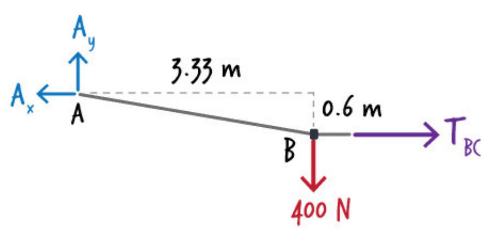


Figure 11: Free-body diagram for left side of cable system

Since only three unknown forces appear on this free-body diagram, we can calculate them using the static equilibrium equations. If we sum the moments about point A, we get the following:

$$\sum M @ A = 0 \quad \Rightarrow 0.6T_{BC} - 400(3.33) = 0 \tag{19}$$

If we solve the above equation for T_{BC} , we find that the tension in segment BC is $T_{BC} = 2220\ N$.

Now, we can write the other two equilibrium equations as follows:

$$\sum F_x = 0 \quad \Rightarrow 2220 - A_x = 0 \qquad [20]$$
$$\sum F_y = 0 \quad \Rightarrow A_y - 400 = 0 \qquad [21]$$

Equations [20] and [21] yield: $A_x = 2220\ N$ and $A_y = 400\ N$.

To determine the tension force in segment *AB*, let's draw the free-body diagram for point *A* (see Figure 12).

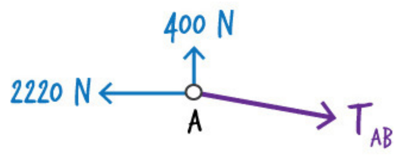


Figure 12: Free-body diagram for point A

Since the sum of the forces at point *A* must be zero, the algebraic sum of the reaction forces must be equal to the tension force in the cable. Therefore, we can write the following equation:

$$\sum F_x = 0 \quad \Rightarrow T_{AB}^2 = 2220^2 + 400^2 \qquad [22]$$

We find that the tension force in segment *AB* is $T_{AB} = 2255.75\ N$.

Because the cable system is symmetrical (see Figure 9), we know that the tension force in segment *CD* is equal to the tension force in segment *AB*. Therefore, we know that the tension force in segment *CD* is equal to $T_{CD} = 2255.75\ N$.

Figure 13 shows the results of the analysis and the tension force in each segment of cable.

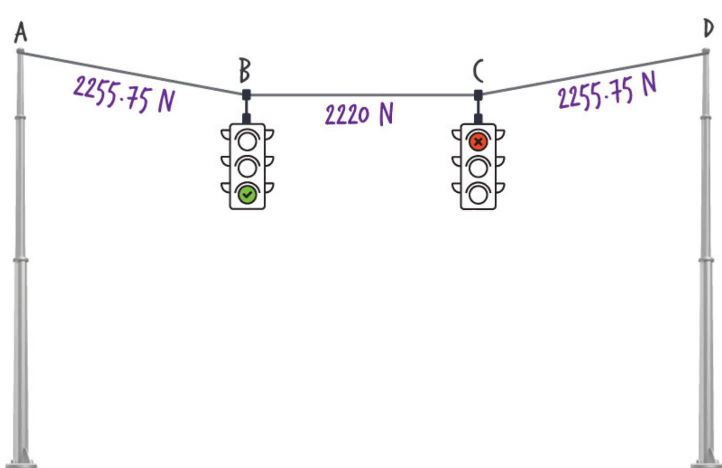


Figure 13: Results of analysis of tension in cables with two equidistant lights

To reinforce the process of analyzing cables subjected to concentrated loads, let's consider a related example. Suppose the right traffic light is positioned 1.8 meters from the right pole, which makes the loading on the cable unsymmetrical (see Figure 14).

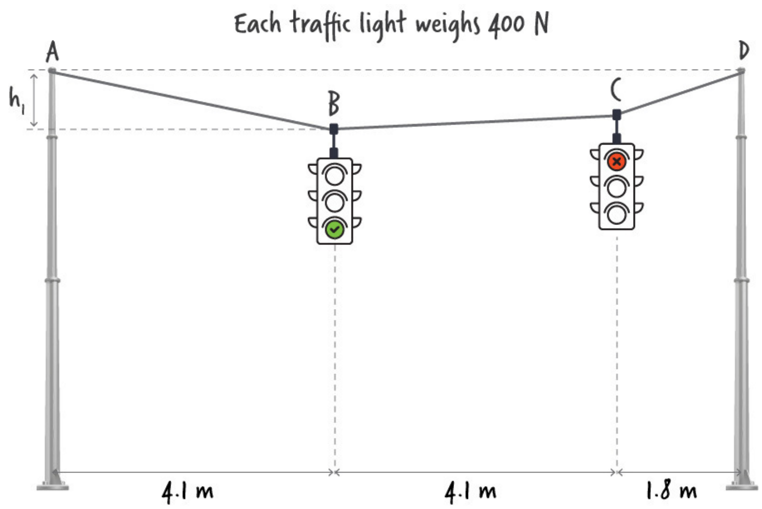


Figure 14: Cables bearing an unsymmetrical load

What are the tension forces in the cable?

We start by drawing the free-body diagram of the entire system (see Figure 15).

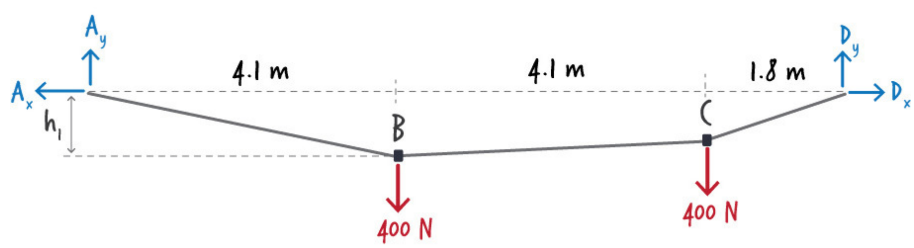


Figure 15: Free-body diagram of cables with unsymmetrical loading

Although we cannot determine all four support reactions, we can calculate the vertical reactions at **A** and **D** using two equilibrium equations.

By summing the moments about point A, we can determine D_y, as shown below.

$$\sum M @ A = 0 \quad \Rightarrow 10D_y - 400(8.2) - 400(4.1) = 0 \quad \Rightarrow D_y = 492\ N \qquad [23]$$

Furthermore, by summing the forces in the **y**-direction, we can calculate A_y as shown in Equation [24].

$$\sum F_y = 0 \quad \Rightarrow A_y + D_y - 400 - 400 = 0 \quad \Rightarrow A_y = 308\ N \qquad [24]$$

To determine the tension force in segment BC, similarly to the previous example, we can draw the free-body diagram for the left segment of the system as depicted in Figure 16.

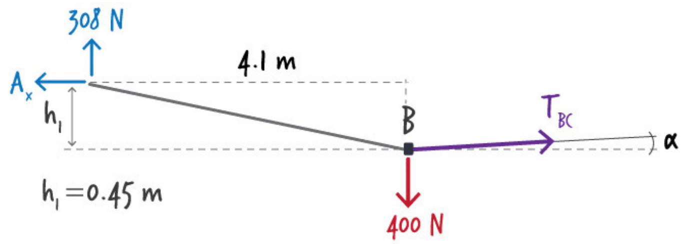


Figure 16: Free-body diagram for the left segment of the system of cables with unsymmetrical loading

The above diagram embodies four unknowns. There are two unknown forces (A_x and T_{BC}), one unknown angle (α), and one unknown distance (h_1). To solve the problem, we need to know one of these unknowns. Let's assume that the vertical distance from point *A* to point *B* is 0.45 meters. That is, $h_1 = 0.45\ m$.

Now, if we sum the moments about point *B*, we can determine A_x as shown in Equation [25].

$$\sum M\ @\ B = 0 \quad \Rightarrow 0.45A_x - 308(4.1) = 0 \quad \Rightarrow A_x = 2806\ N \qquad [25]$$

Then, summing the forces in the *x* and *y* directions, we can determine the *x*- and *y*-components of T_{BC} as follows:

$$\sum F_x = T_{BC}\cos\alpha - 2806 = 0 \quad \Rightarrow T_{BC}\cos\alpha = 2806\ N \qquad [26]$$
$$\sum F_y = T_{BC}\sin\alpha + 308 - 400 = 0 \quad \Rightarrow T_{BC}\sin\alpha = 92\ N \qquad [27]$$

Knowing the *x*- and *y*-components of T_{BC} , we can determine the tension in segment *BC* (see Equation [28]).

$$T_{BC}^2 = (T_{BC}\cos\alpha)^2 + (T_{BC}\sin\alpha)^2 = 2806^2 + 92^2 \quad \Rightarrow T_{BC} = 2807\ N \qquad [28]$$

Similarly to the previous example, to determine the tension force in segment *AB*, we can use the free-body diagram of point *A* (see Figure 17).

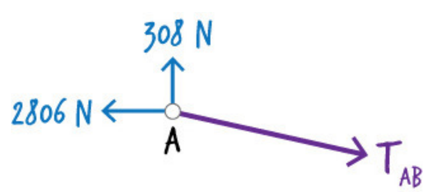


Figure 17: Free-body diagram of point A

The sum of the support reactions at *A* must be equal to T_{AB} . Knowing this fact, we can write the following:

$$T_{AB}^2 = 308^2 + 2806^2 \quad \Rightarrow T_{AB} = 2823\ N \qquad [29]$$

To determine the tension force in segment CD, let's draw the free-body diagram for point D.

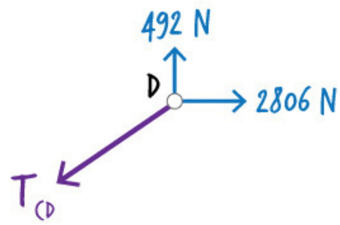


Figure 18: Free-body diagram for point D

Since the sum of the forces at D must be zero, we can write the following:

$$T_{CD}^2 = 492^2 + 2806^2 \quad \Rightarrow T_{CD} = 2849\ N \qquad [30]$$

The results of our analysis and the calculated tension forces in the cable are written next to each segment in Figure 19.

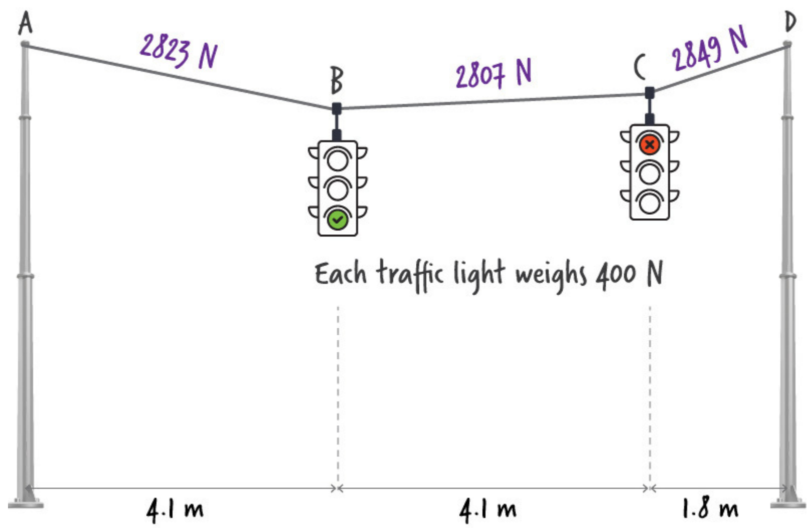
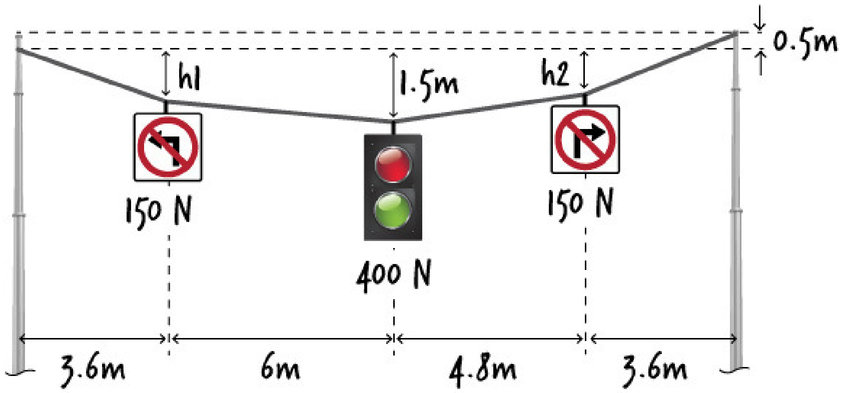


Figure 19: Results of analysis of cables with unsymmetrical loading

As we demonstrated in this lecture, when subjected to concentrated loads only, the analysis of cable systems involves applying the static equilibrium equations to different segments of the system while keeping in mind that no more than three unknowns should be present in any segment.

We will continue our discussion on cables in the next lecture. For now, see if you can solve the following problems.

Problem A: Determine heights h_1 and h_2 in the cable system shown below.



Problem B: Given that tension in segment *AB* is 130 N, determine weight *w*.