Structural Analysis

Lecture Series

SA57: Reaction Influence Line for Continuous Beams

This document is a written version of video lecture SA57 which can be found online at the web addresses listed below

Educative Technologies, LLC

Lab101.Space
https://www.youtube.com/c/drstructure

2018

Structural Analysis – SA57
(Reaction Influence Line for Continuous Beams)

This lecture examines the procedure for constructing reaction influence lines for continuous beams.

Consider a two-span continuous beam with vertical reactions at A, B, and C, as shown in Figure 1. We can qualitatively draw each reaction influence line with ease. Similar to the procedure for a determinate beam, we push the beam upward by a unit force at the support and draw the resulting deformation. In doing so, we assume the beam is no longer restrained by the support and can move upwards. Figure 1 shows the influence lines for the reaction forces at A, B, and C. Note that unlike the case of determinate beams, the resulting deformation of a continuous beam is nonlinear.

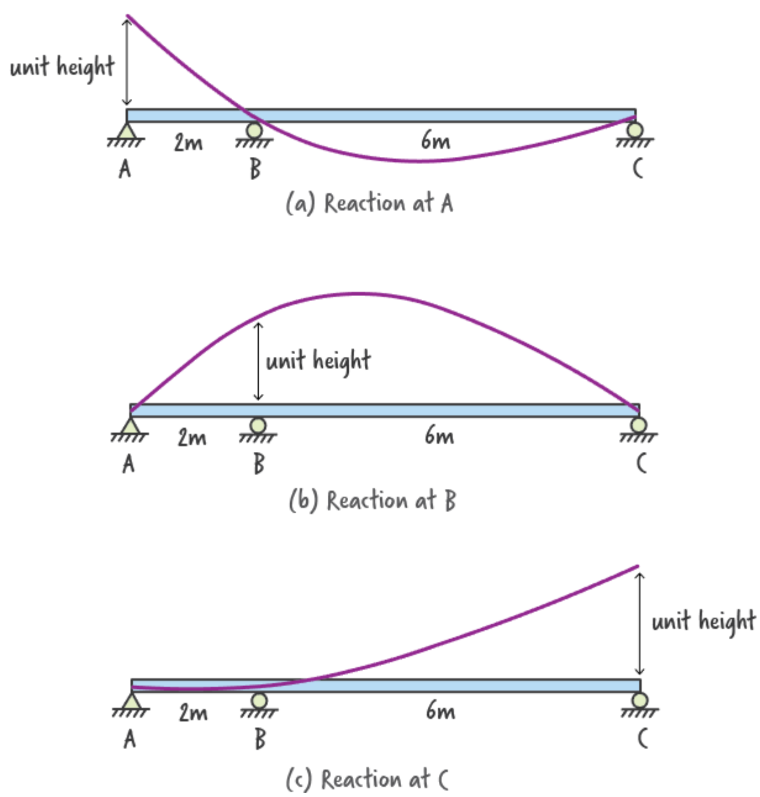


Figure 1: Reaction influence lines for the beam

Diagrams are an excellent way to view influence lines qualitatively; however, they are of limited use for analyzing the effects of a concentrated moving load on a beam. Let's look at why this may be the case.

Suppose we wish to determine the maximum reaction force at B due to a moving load. The influence line shows that the maximum reaction at B occurs when the load is positioned at D (see Figure 2). But where is the exact location of D? Because the diagram is qualitatively drawn, it does not provide this information. To determine the location of the maximum force, we need to derive a mathematical equation for representing the influence line.

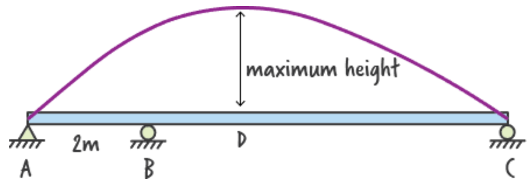


Figure 2: Location of the maximum height of the influence line for the reaction force at B

Similarly, if we want to determine the maximum downward (negative) reaction force at A due to a concentrated moving load, we need an equation for the influence line. The position of the lowest point on the diagram cannot be ascertained from the diagram alone (See Figure 3).

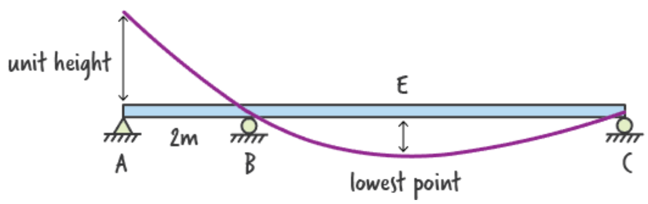


Figure 3: Location of the minimum height of the influence for the reaction force at A

So, how do we write a mathematical expression for a reaction influence line?

It's simple! We must first parametrically analyze the beam using, for example, the slope-deflection method. If you are unfamiliar with this method, please review Lectures SA27 through SA33. You can find the video lectures on the Lab101.space website.

Let's denote the position of the unit load relative to the left end of the beam as n, where $0 \leq n \leq 8$ (see Figure 4).

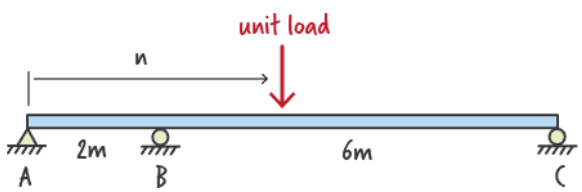


Figure 4: A two-span continuous beam subjected to a unit load

The beam consists of two spans: AB and BC. Therefore, we must write a pair of slope-deflection equations for each span. For Span AB, these are:

$$M_{AB} = \frac{2EI}{2}(2\theta_A + \theta_B) + FEM_{AB} \quad [1]$$

$$M_{BA} = \frac{2EI}{2}(\theta_A + 2\theta_B) + FEM_{BA} \quad [2]$$

where θ_A and θ_B are the end rotations, and M_{AB} and M_{BA} are the end moments for Span AB (see Figure 5).

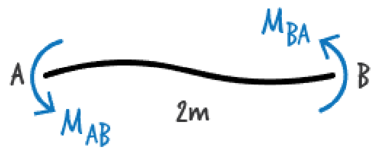


Figure 5: Member-end moments for AB

For the fixed-end moments (FEM) in the above equations, a concentrated load acting on the beam can lead to two scenarios.

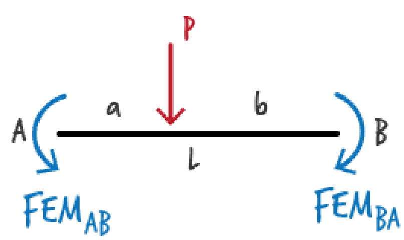
Scenario 1: When $n \geq 2$, $FEM_{AB} = FEM_{BA} = 0$. This is true when the load acts not on Span AB but Span BC.

Scenario 2: When $n \leq 2$, the fixed-end moments can be computed using the following equations:

$$FEM_{AB} = \frac{n(2-n)^2}{4} \quad [3]$$

$$FEM_{BA} = \frac{-(2-n)n^2}{4} \quad [4]$$

A graphical representation of the fixed-end moments is shown in Figure 6.



$$FEM_{AB} = \frac{Pab^2}{L^2} \qquad FEM_{BA} = \frac{Pa^2b}{L^2}$$

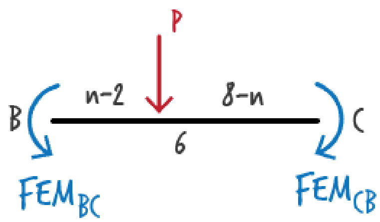
Figure 6: Fixed-end moment equations for Span AB

We also need to write a pair of slope-deflection equations for Span BC, as follows:

$$M_{BC} = \frac{2EI}{6}(2\theta_B + \theta c) + FEM_{BC} \quad [5]$$

$$M_{CB} = \frac{2EI}{6}(\theta_B + 2\theta c) + FEM_{CB} \quad [6]$$

For this span, the fixed-end moments are shown in Figure 7.



$$FEM_{BC} = \frac{1}{36}(n-2)(8-n)^2 \qquad\qquad FEM_{CB} = \frac{-1}{36}(n-2)^2(8-n)$$

Figure 7: Fixed-end moment equations for Span BC

The member-end moments for BC are shown in Figure 8.

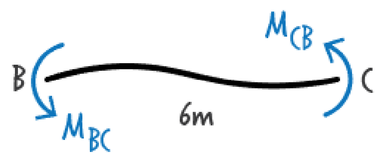


Figure 8: Member-end moments for BC

Similar to Span AB, when $n \leq 2$, the fixed-end moments in Equations [5] and [6] are zero. Otherwise, we must use the expressions given in Figure 7.

Since the beam has three support joints, we need to write three (3) moment equilibrium equations, one per support (see Figure 9).

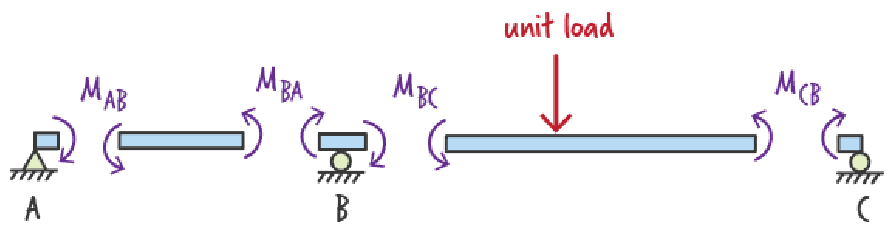


Figure 9: Member-end moments at the supports

The equilibrium equations are:

$$\Sigma M_{@A} = M_{AB} = 0 \qquad [7]$$

$$\Sigma M_{@B} = M_{BA} + M_{BC} = 0 \qquad [8]$$

$$\Sigma M_{@C} = M_{CB} = 0 \qquad [9]$$

When expanded, these equations can be written as two distinct sets of equations that depend on the value of n. For $n \leq 2$, we obtain:

$$\frac{2EI}{2}(2\theta_A+\theta_B)+\frac{n(2-n)^2}{4}=0 \qquad [10]$$

$$\frac{2EI}{2}(\theta_A+2\theta_B)-\frac{(2-n)n^2}{4}+\frac{2EI}{6}(2\theta_B+\theta_C)=0 \qquad [11]$$

$$\frac{2EI}{6}(\theta_B+2\theta_C)=0 \qquad [12]$$

For $n \geq 2$, we obtain:

$$\frac{2EI}{2}(2\theta_A+\theta_B)=0 \qquad [13]$$

$$\frac{2EI}{2}(\theta_A+2\theta_B)+\frac{2EI}{6}(2\theta_B+\theta_C)+\frac{1}{36}(n-2)(8-n)^2=0 \qquad [14]$$

$$\frac{2EI}{6}(\theta_B+2\theta_C)-\frac{1}{36}(n-2)^2(8-n)=0 \qquad [15]$$

The solution to the first set of equations is:

$$\begin{Bmatrix}\theta_A\\ \theta_B\\ \theta_c\end{Bmatrix}=\frac{n}{32EI}\begin{Bmatrix}(n-2)(10-3n)\\ 2(4-n^2)\\ n^2-4\end{Bmatrix} \qquad n\leq 2 \qquad [16]$$

For the second set, we get:

$$\begin{Bmatrix}\theta_A\\ \theta_B\\ \theta_C\end{Bmatrix}=\frac{n-2}{288EI}\begin{Bmatrix}112-22n+n^2\\ 2(-112+22n-n^2)\\ -(80-98n+11n^2)\end{Bmatrix} \qquad n\geq 2 \qquad [17]$$

Substituting these solutions back into the slope-deflection equations, we define the member-end moments. For Set 1, we obtain:

$$\begin{Bmatrix}M_{AB}\\ M_{BA}\\ M_{BC}\\ M_{CB}\end{Bmatrix}=\frac{n}{32}\begin{Bmatrix}0\\ n^2-4\\ 4-n^2\\ 0\end{Bmatrix} \qquad n\leq 2 \qquad [18]$$

For Set 2, we have:

$$\begin{Bmatrix}M_{AB}\\ M_{BA}\\ M_{BC}\\ M_{CB}\end{Bmatrix}=\frac{(2-n)}{96}\begin{Bmatrix}0\\ 112-22n+n^2\\ -112+22n-n^2\\ 0\end{Bmatrix} \qquad n\geq 2 \qquad [19]$$

Now that we know the member-end moments, we can easily determine the reaction forces at A and C.

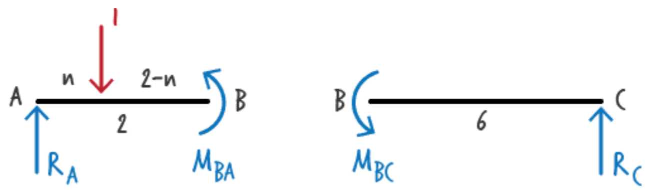


Figure 10: Free-body diagram of the beam segments for Equation Set 1

The two free-body diagrams presented in Figure 10 enable us to write the following equilibrium equations:

$$M_{BA} + 1(2 - n) - 2R_A = 0 \quad [20]$$
$$M_{BC} + 6R_C = 0 \quad [21]$$

Solving for R_A and R_C, we get:

$$R_A = (n^3 - 36n + 64)/64 \quad n \leq 2 \quad [22]$$
$$R_c = (n^3 - 4n)/192 \quad n \leq 2 \quad [23]$$

Moreover, since the sum of the forces for the entire beam must vanish in the y-direction, we can write:

$$R_B = 1 - R_A - R_C \quad [24]$$
$$R_B = (28n - n^3)\ /\ 48 \quad n \leq 2 \quad [25]$$

We need to develop another set of algebraic equations for the support reactions based on the second solution. Figure 11 shows the free-body diagrams of the two beam spans with the unit load on the right span.

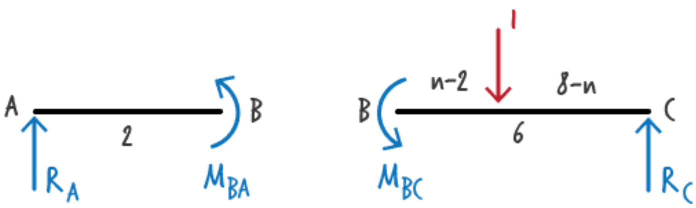


Figure 11: Free-body diagram of the beam segments for Equation Set 2

The following equilibrium equations can be used to yield the support reactions at A and C:

$$M_{BA} - 2R_A = 0 \quad [26]$$
$$M_{BC} + 6R_C - 1(n - 2) = 0 \quad [27]$$

Solving the above equations for R_A and R_C, we obtain:

$$R_A = (2 - n)(n^2 - 22n + 112)/192 \quad n \geq 2 \quad [28]$$
$$R_c = (-n^3 + 24n^2 - 60n + 32)/576 \quad n \geq 2 \quad [29]$$

Then we can write:

$$R_B = 1 - R_A - R_C \quad [30]$$

Or,

$$R_B = (n^3 - 24n^2 + 132n - 32) \;/\; 144 \qquad [31]$$

In summary, the support reactions can be expressed in terms of the load moving load (n) as

$$R_A = \begin{cases} (n^3 - 36n + 64)/64 & n \le 2 \\ (2-n)(n^2 - 22n + 112)/192 & n \ge 2 \end{cases} \qquad [32]$$

$$R_B = \begin{cases} (28n - n^3) \;/\; 48 & n \le 2 \\ (n^3 - 24n^2 + 132n - 32)/144 & n \ge 2 \end{cases} \qquad [33]$$

$$R_C = \begin{cases} (n^3 - 4n) \;/\; 192 & n \le 2 \\ (-n^3 + 24n^2 - 60n + 32)/576 & n \ge 2 \end{cases} \qquad [34]$$

Now we are ready to determine the maximum effect of a moving load on the reactions.

Suppose our beam is a part of a bridge that must be able to carry a maximum vehicular load of 20 kN. We must determine the maximum upward and downward reaction forces that could develop at A (see Figure 12).

Observing the influence line for the reaction force, we can see that the maximum upward reaction occurs at A when the concentrated load is acting at A (see Figure 12). The magnitude of the reaction force is $1 \times 20 = 20\text{ kN}$.

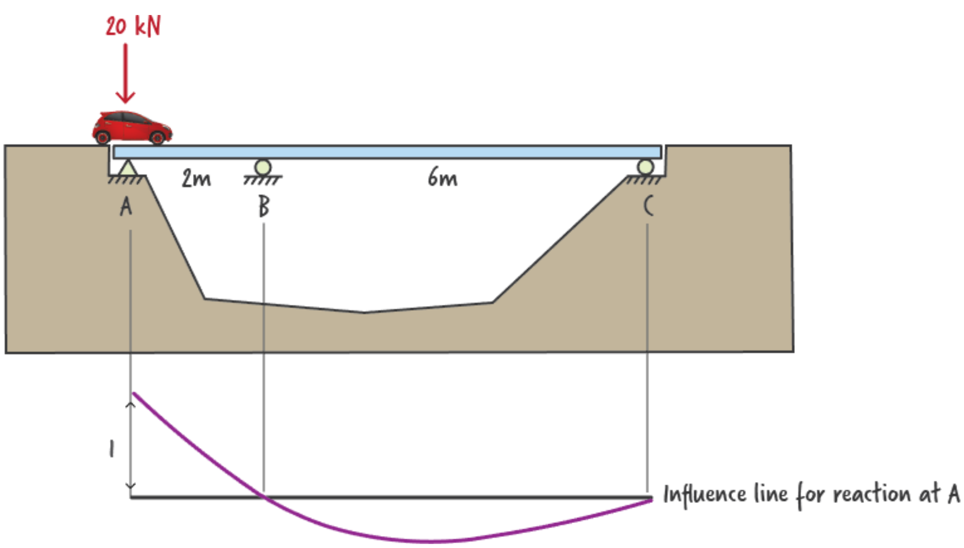


Figure 12: Load position for creating the maximum upward reaction at A

To determine the position of the load that causes the maximum downward reaction to develop at A, we need to locate the position on the diagram where the minimum reaction occurs. By visually inspecting the diagram, we see that the position is located between B and C and therefore governed by the second reaction equation. Thus, taking the derivative of the equation and setting it equal to zero, we can find the position that minimizes the function as follows:

$$\frac{d}{dn}(2-n)(n^2 - 22n + 112)/192 = \frac{1}{64}(-52 + 16n - n^2) = 0 \qquad [35]$$

Solving Equation [35] for n, we get $n = 8 - 2\sqrt{3} \approx 4.53$.

The function for reaction at A (Equation [32]) evaluated at $n = 4.53$ yields

$$R_A = -\frac{\sqrt{3}}{4} \approx -0.43$$

Therefore, the maximum downward (negative) reaction force at A equals $0.43(20) = -8.66\ \text{kN}$ (see Figure 13).

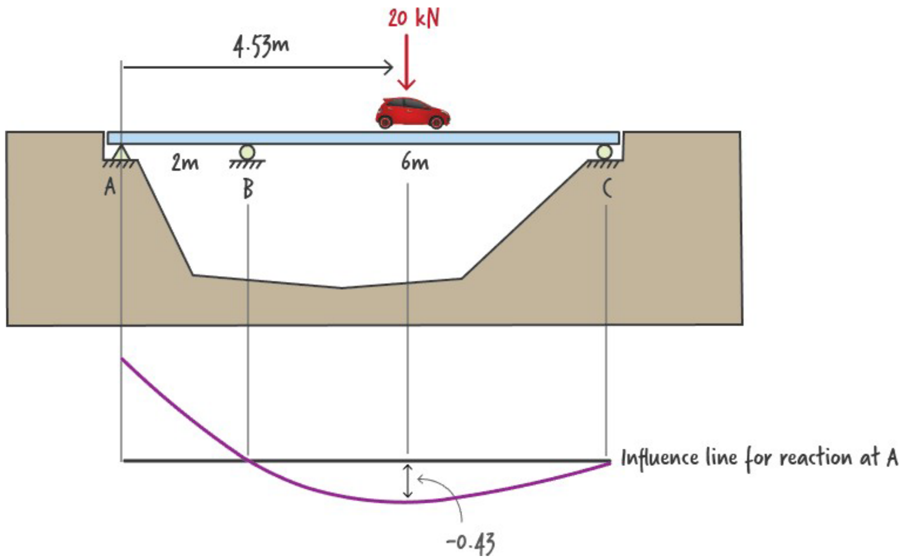


Figure 13: Load position for creating the maximum downward reaction at A

We also need to find the maximum upward reaction at B. The maximum height of the influence line is located in Segment BC, as shown in Figure 14. This means that we need to take the derivative of the second reaction equation, set it equal to zero and solve for n, as follows:

$$\frac{d}{dn}(n^3 - 24n^2 + 132n - 32)\ /\ 144 = \frac{1}{48}(44 - 16n + n^2) = 0 \qquad [36]$$

Equation [36] yields $n = 8 - 2\sqrt{5} \approx 3.53$. Equation [33] evaluated at $n = 3.53$ yields:

$$R_B = \frac{5\sqrt{5}}{9} \approx 1.24$$

Therefore, the maximum upward reaction at B is $1.24(20) = 24.85\ \text{kN}$ (see Figure 14).

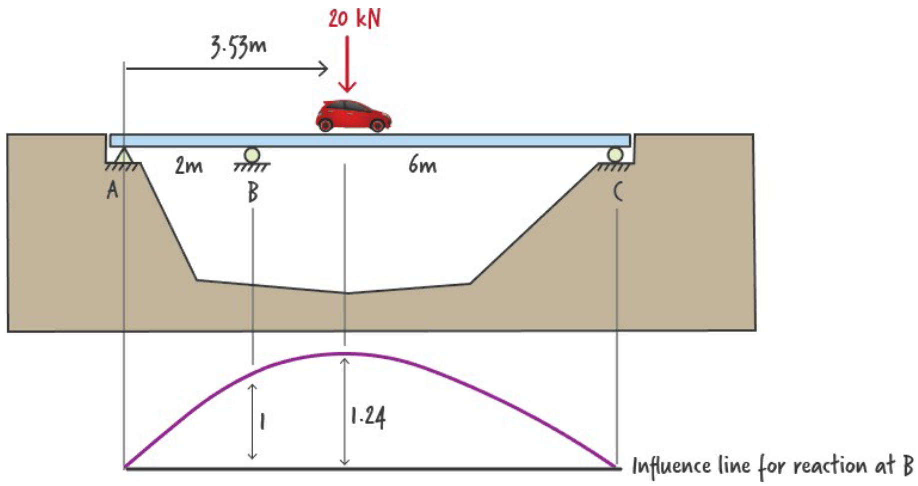


Figure 14: Load position for creating the maximum upward reaction at B

If you want to see how the influence lines change shape as the position of the roller support at B changes, go to the interactive webpage referenced below.

https://Lab101.Space/iexamples/SA57/index.html