Structural Analysis
Lecture Series

SA12: Deflection in Beams
The Double Integration Method

This document is a written version of video lecture SA12, which can be found online at the web addresses listed below.

Educative Technologies, LLC

http://www.Lab101.Space
https://www.youtube.com/c/drstructure

The contributions of Galina Jorgic in preparing this document are gratefully acknowledged.

2019

Structural Analysis – SA12
Deflection in Beams - The Double Integration Method

This lecture describes the use of the double integration method for calculating deflection in beams. Given a statically determinate beam, the method enables us to derive an algebraic equation that describes its deformed shape.

Figure 1 shows a simply supported beam subjected to a generalized load. The deformed shape of the beam, called its elastic curve, is shown using dashed lines.

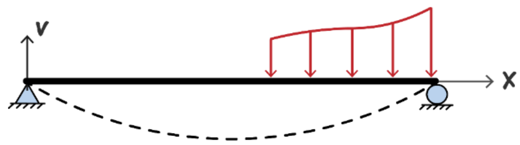

Figure 1: A simply supported beam and its elastic curve

Let's denote the beam's deflection as **v**. Since the deflection varies along the x-axis, **v** is a function of x.

The beam rests on a pin and a roller at its ends, therefore, **v** evaluates to zero at the end points. However, its values are not known within the length of the beam. Here, our objective is to determine an equation for **v** that enables us to calculate the deflection at any point along the length of the beam.

We start the derivation by drawing a tangent line to the elastic curve at an arbitrary point along the x-axis, as shown in Figure 2.

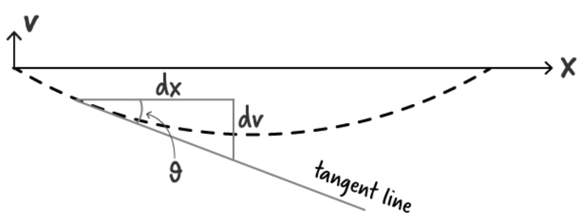

Figure 2: The slope of an elastic curve at an arbitrary point

The tangent line forms a right triangle with the horizontal and vertical axes. In the figure above, the sides of the triangle are labeled dx and d**v**. Therefore, the angle that the tangent line makes with the horizontal axis, denoted by θ, can be defined in the following manner.

$$\tan(\theta) = d\mathbf{v}/dx \qquad [1]$$

Since deflections in a typical beam are relatively small, we can assume that θ is a small angle. Therefore, we can equate the tangent of θ to the angle itself. That is, if θ is given in radians, we can write: $\tan(\theta) = \theta$.

Consequently, Equation [1] can be rewritten as follows.

$$\tan(\theta) = d\mathbf{v}/dx = \theta \qquad [2]$$

Or,

$$d\mathbf{v} = \theta\, dx \qquad [3]$$

By integrating both sides of Equation [3], we can obtain **v** in an integral form, as shown below.

$$\int d\mathbf{v} = \int \theta\, dx \quad \Rightarrow \mathbf{v} = \int \theta\, dx \qquad [4]$$

To complete our derivation, we need to integrate the right side of Equation [4]. However, before doing so, we must express θ in terms of x. This can be done by writing θ in terms of the beam's bending moment equation, which is written in terms of x.

To relate θ to the beam's bending moment, let's visualize the deformed shape of the beam as shown below.

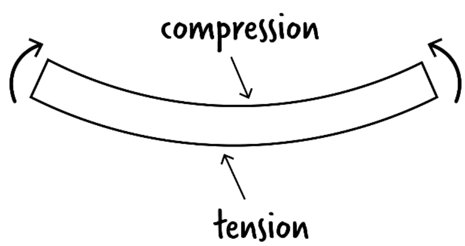

Figure 3: The deformed shape of a beam segment due to a positive bending moment

When the beam is subjected to a positive bending moment, its top fibers undergo compression while the bottom fibers experience tension. As depicted in Figure 4, the compression and tension zones are naturally separated along a plane located in the middle part of the beam.

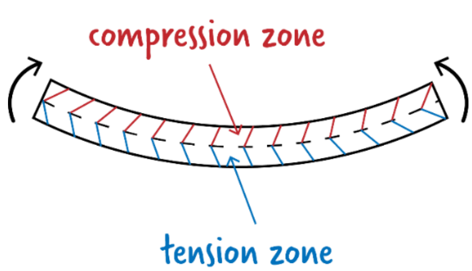

Figure 4: The compression and tension zones in a typical beam segment

Figure 5 shows two neighboring planes in the compression zone of a beam segment. They are labeled plane A and plane B. Since plane A is at the outer surface, its fibers are being compressed more than the fibers on plane B, which is located below plane A. The amount of compression in a fiber reduces in relation to its proximity to the middle of the beam. The closer the fiber is to the middle of the beam, the less compression it experiences.

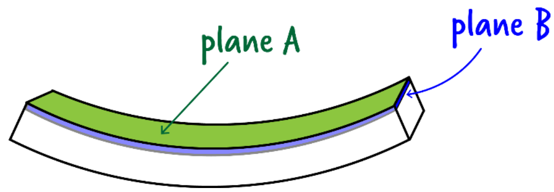

Figure 5: Two compression planes in a beam segment

Beam fibers in the tension zone behave in a similar manner. The fibers closer to the middle part of the beam are going to be elongated less than the fibers closer to the surface. Therefore, there must be a surface inside the beam where the fibers are neither compressed nor elongated. This surface, which separates the tension from the compression zone, is called the neutral surface (see Figure 6).

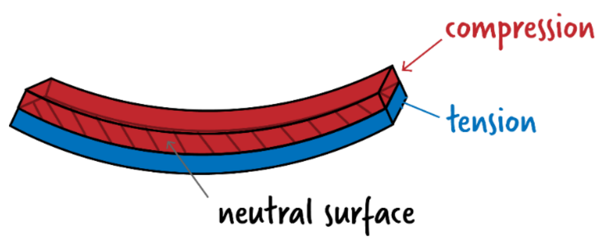

Figure 6: The neutral surface in a beam

When a beam is being examined in the two-dimensional space, we refer to the neutral surface as the neutral axis. Figure 7 depicts a two-dimensional beam and its neutral axis. When a beam bends, all of its fibers change length, except for the fibers along its neutral axis.

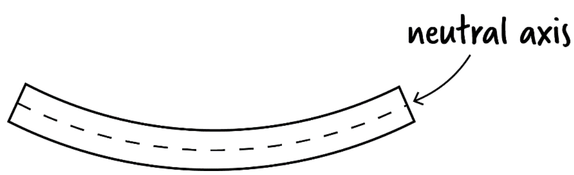

Figure 7: The neutral axis in a beam

Consider a thin slice of a beam having a width denoted by dx, as depicted below.

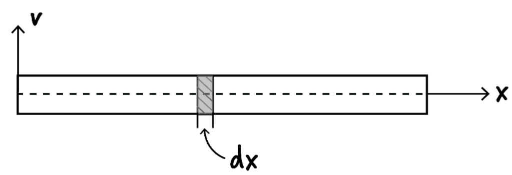

Figure 8: A thin slice of a beam

Figure 9 represents the deformed shape of the thin slice caused by a positive bending moment.

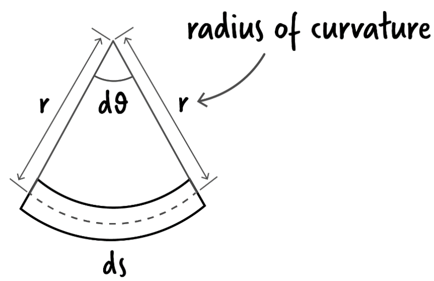

Figure 9: A deformed beam segment due to a positive bending moment

Let's refer to the arc length along the neutral axis of the beam slice as ds. The arc can be assumed to be that of a circle having r as its radius. We refer to r as the radius of curvature. From basic geometry, we can write the following relationship relating r, ds, and $d\theta$.

$$r\,d\theta = ds \qquad [5]$$

Note that $d\theta$ is the difference between the end slopes θ_1 and θ_2, as illustrated in the figure below.

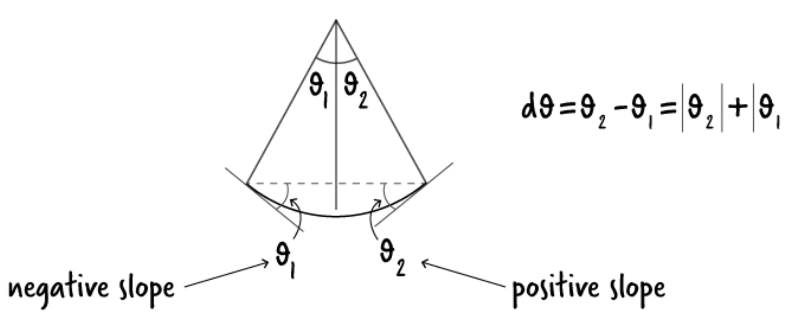

Figure 10: The relationship between $d\theta$ and the end slopes in a beam segment

Since length ds is along the beam's neutral axis, it equals dx (ds = dx). Therefore, Equation [5] can be rewritten as Equation [6].

$$r\,d\theta = dx \tag{6}$$

Or:

$$d\theta = \frac{1}{r}dx \tag{7}$$

Integrating both sides of Equation [7], the following equation results.

$$\theta = \int \frac{1}{r}\,dx \tag{8}$$

Equation [8] represents θ in terms of the radius of curvature (r). Now we need to write r in terms of the bending moment (M).

The amount of deflection in a beam is directly proportional to the magnitude of the bending moment. The larger the bending moment, the larger the deflection. Furthermore, the radius of curvature is inversely proportional to the magnitude of the bending moment. That is, as the bending moment becomes larger and larger, the radius of curvature gets smaller and smaller, as illustrated below.

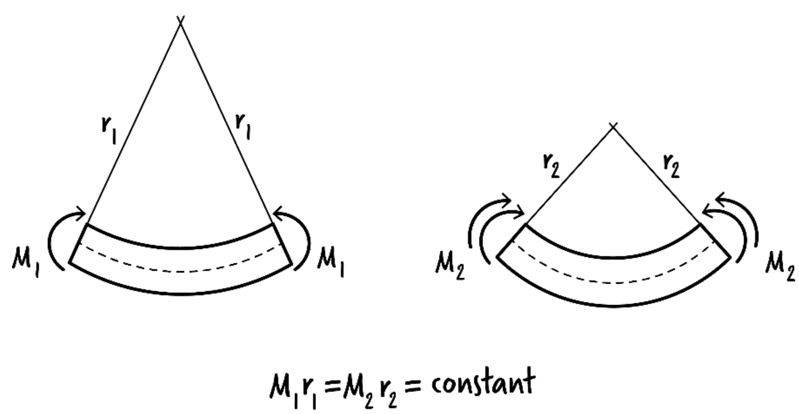

Figure 11: The relationship between the bending moment and radius of curvature in a beam

For linear elastic material, the product of M and r remains constant. This constant is a function of the geometry and material properties of the beam. More specifically, $M\,r = E\,I$ where E is the modulus of elasticity of the material and I is the moment of inertia of the cross-section of the beam about the axis of bending (see Figure 12).

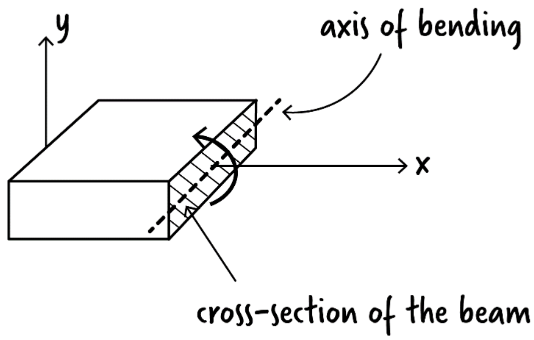

Figure 12: Axis of bending in a typical beam

We can express the relationship between r, M, E, and I using the following equation.

$$\frac{1}{r}=\frac{M}{EI} \qquad [9]$$

Then, Equation [9] can be used to rewrite Equation [8].

$$\theta=\int\frac{1}{r}\,dx \quad \Rightarrow \theta=\int\frac{M}{EI}\,dx \qquad [10]$$

And since $\mathbf{v}=\int\theta\,dx$ (see Equation [4]), Equation [10] becomes:

$$\mathbf{v}=\iint\frac{M}{EI}\,d^2x \qquad [11]$$

Given that the bending moment (M) is an algebraic expression in terms of x, Equation [11] gives **v** in term of x. Let's see how.

Consider a cantilever beam subjected to a concentrated load of P at its free end. The beam's length is L. We wish to formulate an equation for the beam's elastic curve.

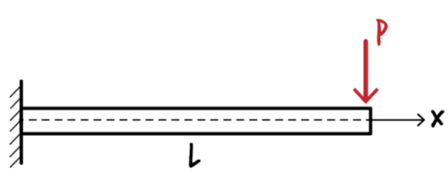

Figure 13: A cantilever beam subjected to a concentrated load

Figure 14 shows the free-body diagram and the bending moment equation for the cantilever beam.

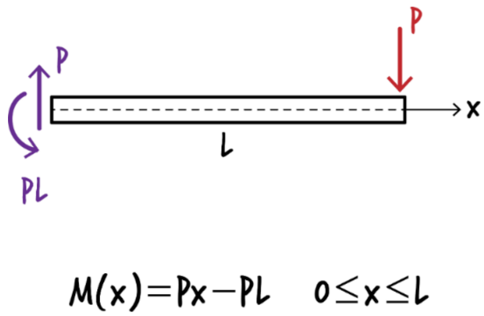

Figure 14: The free-body diagram for a cantilever beam

Therefore, Equation [11] can be expanded as follows.

$$v(x)=\iint \frac{M}{EI}\,d^2x=\frac{1}{EI}\iint (Px-PL)\,d^2x \qquad [12]$$

Note that the right side of the equation is a double integral expression. If we integrate the expression once, we get:

$$v(x)=\frac{1}{EI}\int (\frac{1}{2}Px^2-PLx+C_1)dx \qquad [13]$$

Now we can obtain the algebraic expression for **v** by integrating Equation [13].

$$v(x)=\frac{1}{EI}(\frac{1}{6}Px^3-\frac{1}{2}PLx^2+C_1x+C_2) \qquad [14]$$

Equation [14] consists of two integration constants: C_1 and C_2. They can be determined using the imposed boundary conditions on the beam. There are two imposed boundary conditions on this particular beam: (1) The beam’s deflection at x = 0 is zero; (2) The slope of the elastic curve at x = 0 is zero.

Using Equation [14], the first boundary condition can be written in the following form.

$$v(0)=\frac{1}{EI}(\frac{1}{6}P\times 0^3-\frac{1}{2}PL\times 0^2+C_1\times 0+C_2)=0 \qquad [15]$$

Solving the above equation for C_2, we get: $C_2 = 0$. To apply the second boundary condition, we need to use the slope equation. Since slope is the first derivative of deflection with respect to x, we can write:

$$\theta(x) = \frac{dv}{dx} = \frac{1}{EI}\left(\frac{1}{2}Px^2 - PLx + C_1\right) \qquad [16]$$

Setting slope to zero at x = 0, we can write:

$$\theta(0) = \frac{1}{EI}\left(\frac{1}{2}P\times 0^2 - PL\times 0 + C_1\right) = 0 \qquad [17]$$

The above equation yields: $C_1 = 0$. Therefore, the deflection equation can be written as:

$$v(x) = \frac{1}{EI}\left(\frac{1}{6}Px^3 - \frac{1}{2}PLx^2\right) \qquad [18]$$

Equation [18] is the algebraic expression for the deflection of the beam shown in Figure 13. The equation is valid for any x value between 0 and L.

Let's solve another problem. Consider the simply supported beam shown in Figure 15. Assuming the beam has a constant EI, we wish to formulate its deflection equation.

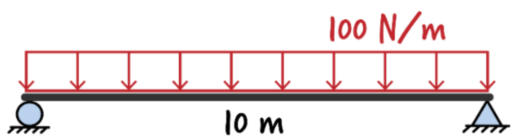

Figure 15: A simply supported beam

The moment equation for the beam is given in the figure below.

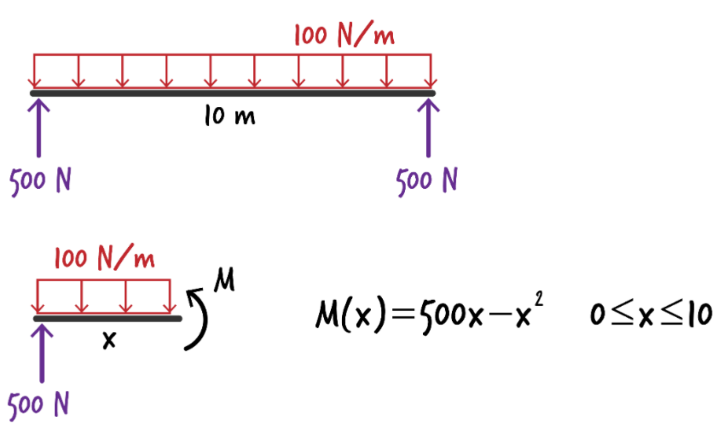

Figure 16: The free-body diagram and bending moment equation for a simply supported beam

Starting from Equation [11], we can express the deflection equation for the beam in the following form.

$$v(x)=\iint \frac{M}{EI}\, d^2x=\frac{1}{EI}\iint (500x-50x^2)\, d^2x \qquad [19]$$

Integrating the right side of Equation [19] with respect to x twice, we obtain the following algebraic expression for **v**.

$$v(x)=\frac{1}{EI}(\frac{250}{3}x^3-\frac{25}{6}x^4+C_1x+C_2) \qquad [20]$$

We now need to apply the boundary conditions in order to determine the integration constants C_1 and C_2. For this beam the boundary conditions are: (1) The deflection of the beam at x = 0 is zero; (2) The deflection of the beam at x = 10 is zero.

The first boundary condition gives us the following expression.

$$v(0)=\frac{1}{EI}(\frac{250}{3}0^3-\frac{25}{6}0^4+C_1\times 0+C_2)=0 \qquad [21]$$

Therefore, $C_2=0$. The second boundary condition yields the following equation.

$$v(10)=\frac{1}{EI}(\frac{250}{3}10^3-\frac{25}{6}10^4+C_1\times 10)=0 \qquad [22]$$

Solving the above equation for C_1, we get: $C_1=\frac{-12500}{3}$. Therefore, the deflection equation for the beam can be written as:

$$v(x)=\frac{25x}{3EI}(10x^2-\frac{x^3}{2}-500) \qquad [23]$$

The above deflection equation is valid for any x value between 0 and 10 meters. We can use it to calculate the deflection at any point along the x-axis, or to graph the beam's elastic curve when the need arises.