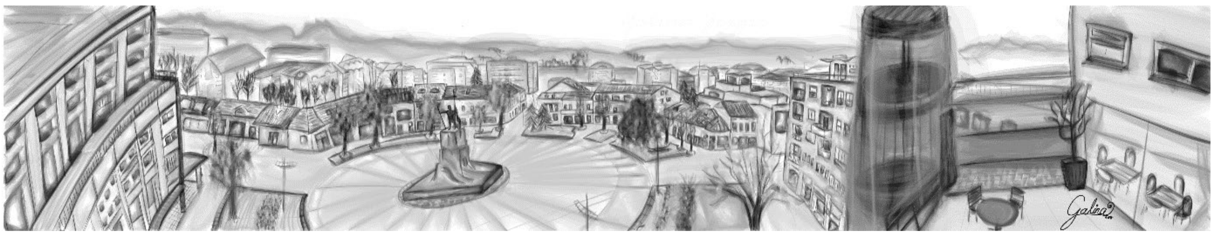
Structural Analysis

Lecture Series



SA59: Calculating Slope and Deflection in Beams Using the Moment-Area Theorems

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Structural Analysis – SA59
The Moment-Area Method for Calculating Slope and Deflection in Beams

In this lecture, we are going to discuss the use of the moment-area method for calculating slope and deflection in beams.

Let's start by considering a cantilever beam subjected to a concentrated moment of 3 kN.m at its free end. The length of the beam is 4 meters.

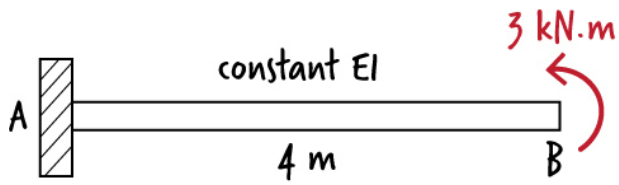


Figure 1: A cantilever beam

Naturally, the beam is going to deflect upward, as shown in Figure 2.

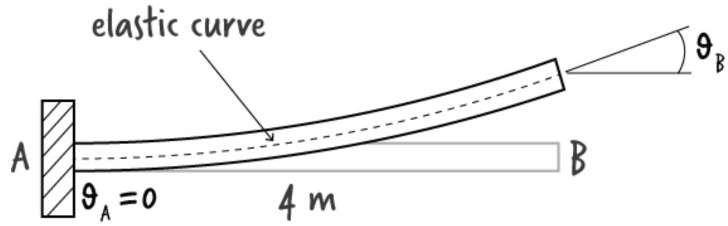


Figure 2: The deformed shape of a cantilever beam subjected to a concentrated moment

We refer to the deformed line along the neutral axis of the beam as its elastic curve. The slope of the curve we denote using ϑ. For example, the slope at A and B are denoted as ϑ_A and ϑ_B, respectively (see Figure 2 above).

According to the moment-area method, the difference between the two slopes is equal to the area under the M/EI diagram between the two points. Expressed mathematically, the theorem can be written as follows:

$$\vartheta_B - \vartheta_A = \int_A^B \frac{M}{EI} dx \quad [1]$$

Please keep in mind that although we are referring to points A and B here, the theorem works for any pair of points on the beam.

Let us see why this relationship holds true.

In Lecture SA22, we showed that for an infinitesimal slice of a beam (see Figure 3), the difference between the slopes of the elastic curve at the ends of the slice can be expressed in terms of the internal bending moment at that point. More precisely, we can write:

$$d\theta = \frac{M}{EI}dx \qquad [2]$$

Where E is the modulus of elasticity of the material and I is the moment of inertia of the beam about the axis of bending. Here we are assuming that the beam has a constant EI.

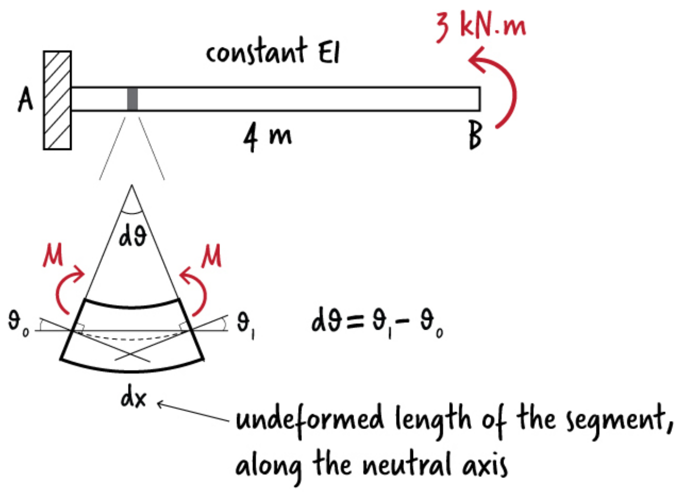


Figure 3: An infinitesimal deformed beam element

Integrating both side of Equation 2 over a specific interval, say from point i to point j along the length of the beam, we get:

$$\int_i^j d\theta = \int_i^j \frac{M}{EI}dx \qquad [3]$$

The left side of the equation equals: $\theta_j - \theta_i$. The right side of the equation is equal to the area under the M/EI diagram between the two points. When we apply the above expression to segment AB of our cantilever beam, Equation 3 turns into Equation 1.

Now that we know the basis for the theorem, let's use it to calculate the rotation at the free end of the beam. The M/EI diagram for the cantilever beam is shown below.

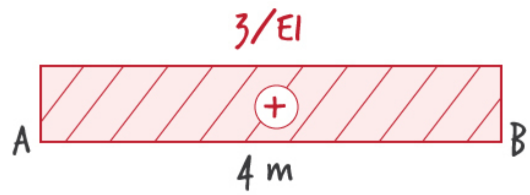


Figure 4: The M/EI diagram for a cantilever beam

Therefore, the area under the diagram between A and B is: $4(3/EI)=12/EI$. Consequently, Equation 1 can be simplified to read:

$$\theta_B - \theta_A = 12/EI \tag{4}$$

And since the slope of the elastic curve at the fixed end of the beam is zero, θ_B can easily be determined as follows:

$$\theta_B - 0 = \theta_B = 12/EI \tag{5}$$

Let's consider another example. Suppose we wish to determine the slope of the elastic curve at the midpoint of the cantilever beam. Since we know θ_A, we can write:

$$\theta_{midpoint} - \theta_A = \int_0^2 \frac{M}{EI} dx \tag{6}$$

The area under the M/EI diagram for the beam segment between 0 and 2 is $2(3/EI)=6/EI$, therefore:

$$\theta_{midpoint} - \theta_A = \theta_{midpoint} - 0 = \theta_{midpoint} = 6/EI \tag{7}$$

Now consider a simply supported beam subjected to a concentrated load at its midpoint, as shown in Figure 5.

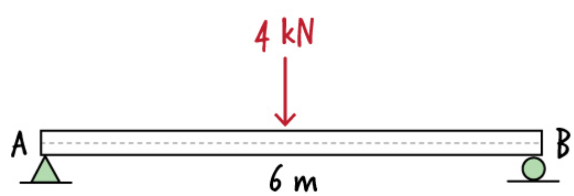


Figure 5: A simply supported beam subjected to a concentrated load

Suppose we wish to determine the slopes of the elastic curve at the ends of the beam. The M/EI diagram and the elastic curve for the entire beam are shown below.

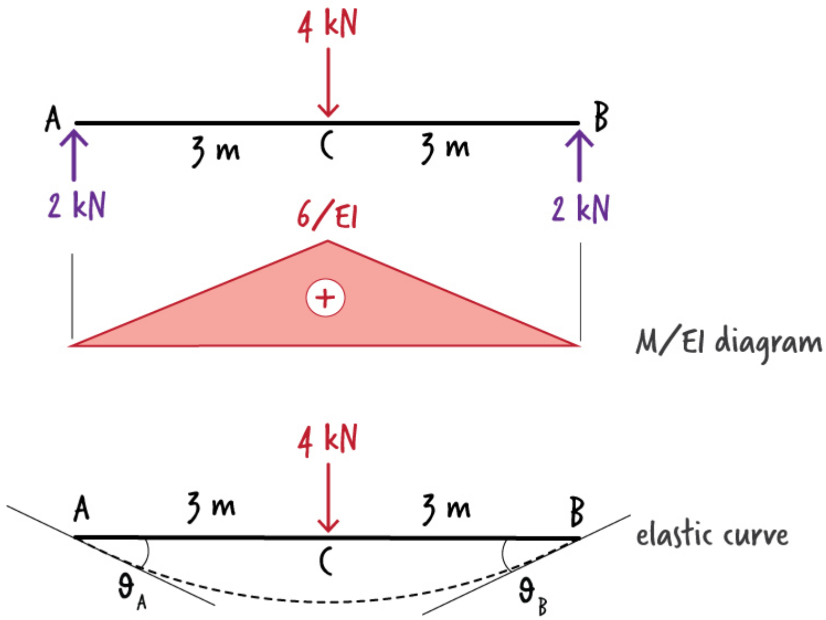


Figure 6: The M/EI and elastic curve diagrams for a simply supported beam

If we label the end rotations of the beam ϑ_A and ϑ_B, knowing that the area under the M/EI diagram is $18/EI$, then according to the theorem we can write:

$$\vartheta_B - \vartheta_A = 6\left(\frac{6}{EI}\right)\left(\frac{1}{2}\right) = 18/EI \quad [8]$$

However, neither ϑ_A nor ϑ_B can be calculated using the above equation, as we have one equation with two unknowns. To be able to use the first moment-area theorem, we need to know one of the slopes in order to determine the other one.

In this case, given the symmetrical nature of the load, we know the slope of the elastic curve at point C where deflection reaches its maximum value. That slope is zero, hence:

$$\vartheta_C - \vartheta_A = 0 - \vartheta_A = \vartheta_A = -3\left(\frac{6}{EI}\right)\left(\frac{1}{2}\right) = -9/EI \quad [9]$$

Knowing ϑ_A, we can now use Equation 8 to determine ϑ_B, as follows:

$$\vartheta_B - \left(-\frac{9}{EI}\right) = 18/EI \tag{10}$$

Or:

$$\vartheta_B = 9/EI \tag{11}$$

As we just saw, the first moment-area theorem has limited computational use. The theorem can be used to calculate the slope of the elastic curve at a point only if we already know the slope at another point. For example, consider the beam shown below.

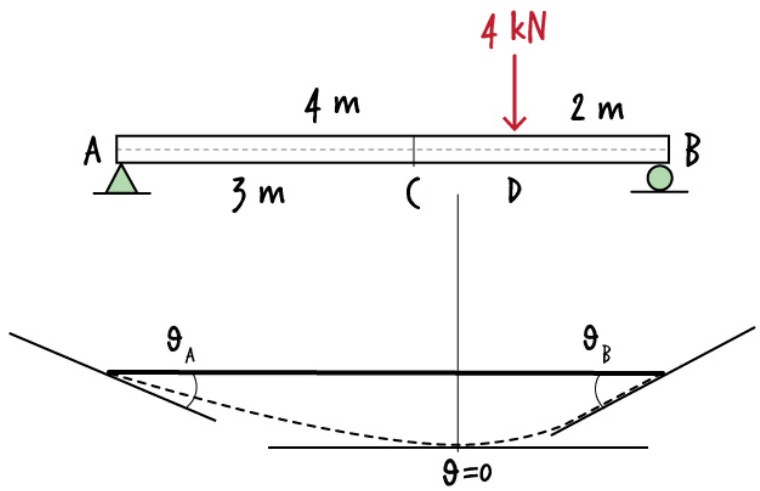


Figure 7: The elastic curve for a beam subjected to an asymmetrical loading

Here the load is not at the center of the beam, and the maximum deflection occurs somewhere between points C and D. Therefore, since the exact location at which ϑ vanishes is not known, the first moment-area method cannot be used to determine the end slopes. For that, we need to utilize the second moment-area theorem.

What is this theorem about? In essence, the second moment-area theorem states that the vertical distance between the tangent lines at A and B, measured at B, is equal to the moment of the M/EI diagram about B.

For our statically determinate beam, the M/EI diagram and the tangent lines at A and B are shown in Figure 8.

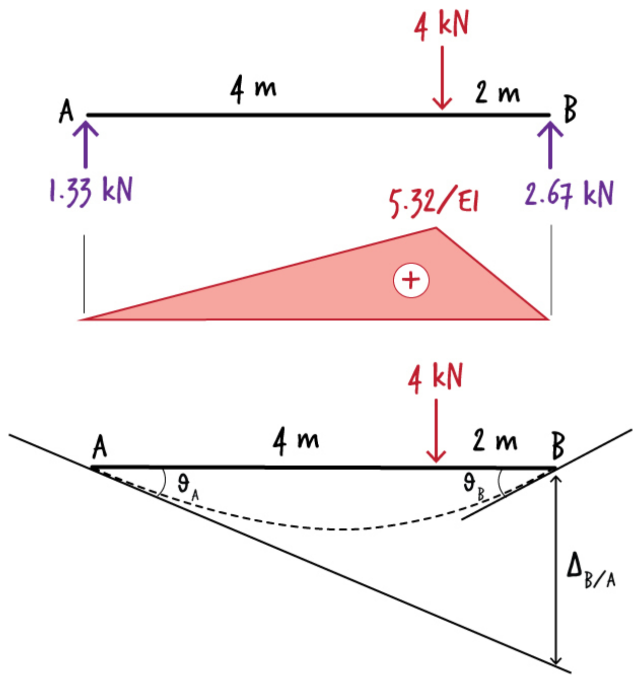


Figure 8: Vertical distance between two tangent lines at the right end of a beam

The vertical distance between the two tangent lines at B is denoted as $\Delta_{B/A}$. According to the second moment-area theorem, this distance is equal to the moment of the M/EI diagram about point B. Or,

$$\Delta_{B/A} = (4)(\frac{5.32}{EI})(\frac{1}{2})(2+\frac{4}{3}) + (2)(\frac{5.32}{EI})(\frac{1}{2})(2\times\frac{2}{3}) = \frac{42.56}{EI} \qquad [12]$$

In writing the above expression, the M/EI diagram is viewed as two right triangles, and the moment of each about B is computed by multiplying the area by the distance from its center to point B. The two distances (moment arms) are shown in blue in the above equation.

Since beam rotations are assumed to be small, we can write:

$$\theta_A \simeq \tan(\theta_A) = \frac{\Delta_{B/A}}{6} \qquad [13]$$

Or

$$\theta_A = \frac{42.56}{6EI} = \frac{7.09}{EI} \qquad [14]$$

Please keep in mind that this is a negative rotation. So, in a strict sense we should write $\vartheta_A = -7.09/EI$. Now that we have ϑ_A, we can use the first moment-area theorem to determine ϑ_B, hence:

$$\vartheta_B - \vartheta_A = \int_0^6 \frac{M}{EI} dx \quad [15]$$

Since the area under the M/EI diagram is: $(5.32/EI)(6)(1/2) = 15.96/EI$, it follows that:

$$\vartheta_B - \vartheta_A = 15.96/EI \quad [16]$$

By substituting $-7.09/EI$ for ϑ_A, we get:

$$\vartheta_B = \frac{15.96}{EI} - \frac{7.09}{EI} = \frac{8.87}{EI} \quad [17]$$

Please note that we can also use the second moment-area theorem to determine ϑ_B. Let's see how. Consider the diagram shown in Figure 9. The height difference between the two tangent lines, measured at A, is labeled $\Delta_{A/B}$.

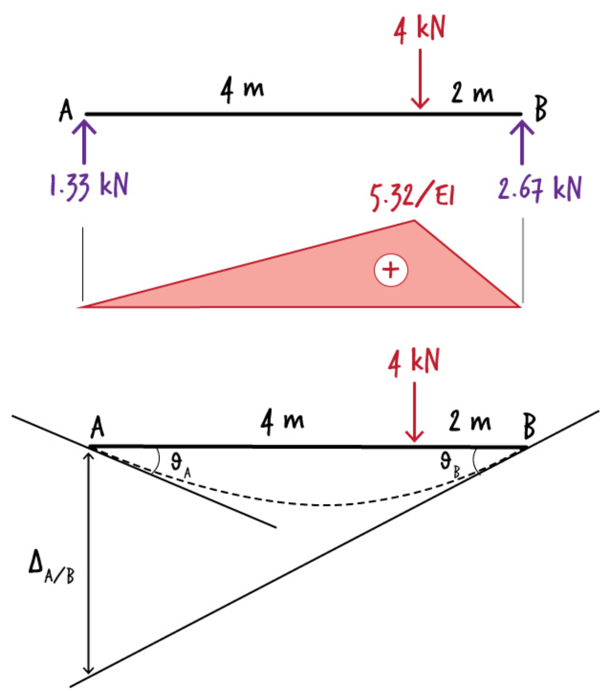


Figure 9: Vertical distance between two tangent lines at the left end of a beam

This height can be determined by taking the moment of the M/EI diagram about point A, as shown below.

$$\Delta_{A/B}=(4)(\frac{5.32}{EI})(\frac{1}{2})(2\times\frac{4}{3})+(2)(\frac{5.32}{EI})(\frac{1}{2})(4+\frac{2}{3})=\frac{53.20}{EI} \quad [18]$$

And since $\Delta_{A/B}=6\theta_B$, it follows that:

$$\theta_B=\frac{53.20}{6EI}=\frac{8.87}{EI} \quad [19]$$

Now that we know how the second moment-area theorem works, let us see why the theorem holds true. Consider an infinitesimal element in our cantilever beam, as depicted in Figure 10.

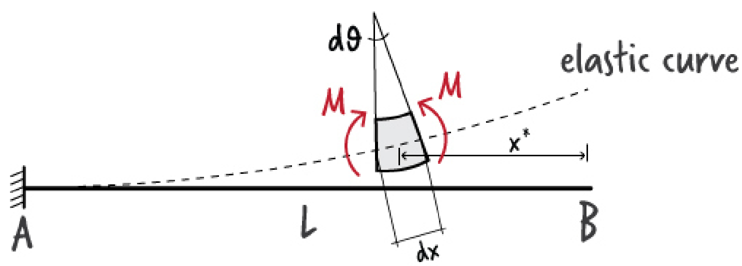


Figure 10: An infinitesimal beam element at an arbitrary position in a beam

The element carries an internal bending moment of M which is causing the differential end slope $d\theta$. The mathematical relationship between M and $d\theta$ is:

$$d\theta=\frac{M}{EI}dx \quad [20]$$

Without loss of generality, let us assume the element is located at distance x^* from the free end of the beam. Multiplying both sides of the above equation by x^*, we get:

$$x^*d\theta=x^*\frac{M}{EI}dx \quad [21]$$

If we integrate both side of the above equation from A to B, we get:

$$\int_A^B x^* d\theta = \int_A^B x^* \frac{M}{EI} dx \quad [22]$$

Figure 11 shows that the left side of Equation 21 equals the vertical distance between the two end tangents at distance x^* from the element.

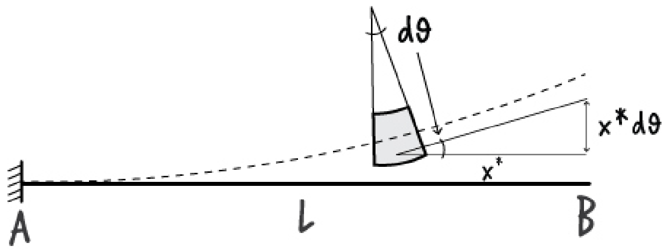


Figure 11: The vertical distance between the end tangent lines of a beam element

When we integrate that distance from A to B, we get the total vertical distance at B between the two tangent lines (lines drawn at A and B), as depicted in figure below.

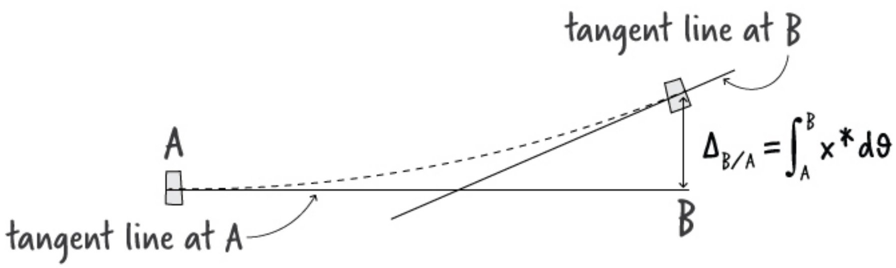


Figure 12: The vertical distance between the tangent lines at two distinct points in a beam

Now that we know the geometric interpretation of the left side of Equation 22, let's turn our attention to the right side of the equation, which represents the first moment of the M/EI diagram about point B. Therefore, we can rewrite it as follows:

$$\Delta_{B/A} = \bar{x} \int_A^B \frac{M}{EI} dx \quad [23]$$

Where $\bar{x}$ is the distance from the center of the M/EI diagram to point B. The above equation represents the second moment-area theorem. Geometrically speaking, it states that the distance between the two tangent lines, measured at B, is equal to the area under the M/EI diagram times the distance from the center of the diagram to point B.

So, for the simply supported beam shown in Figure 7, vertical distance $\Delta_{B/A}$ (as depicted in Figure 8) is computed by taking the moment of the area under the M/EI diagram about point B, and vertical distance $\Delta_{A/B}$ (see Figure 9) is calculated by taking the moment of the same area about point A.

Generally speaking, the second moment-area theorem enables us to determine both slope and deflection in beams, whereas the first moment-area theorem can be used for calculating slopes only.

Let us determine the vertical deflection of the cantilever beam given in Figure 1 using this method. The M/EI diagram and the elastic curve for the beam are shown in Figure 13.

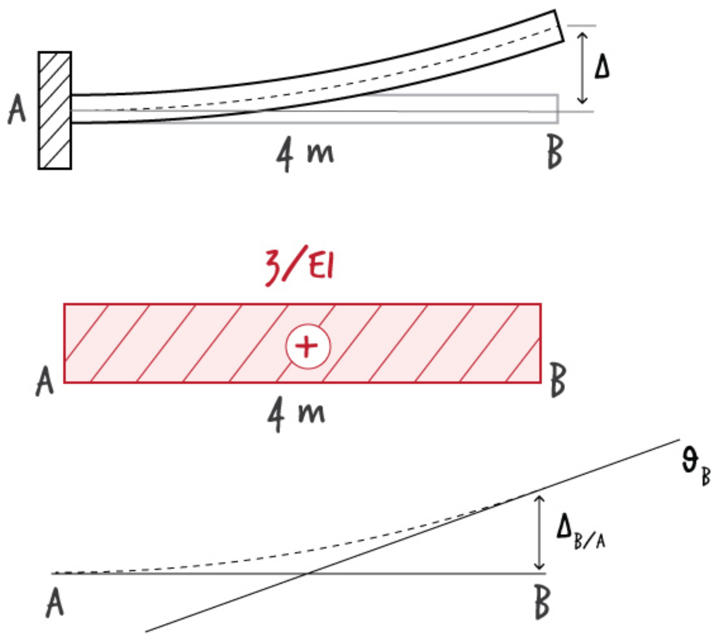


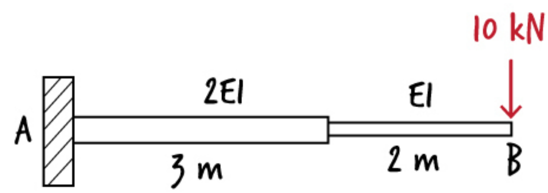
Figure 13: The M/EI diagram and the elastic curve for a cantilever beam

The vertical deflection at B, which is the same as $\Delta_{B/A}$, can be computed by taking the moment of the area under the M/EI diagram (from A to B) about B.

$$\Delta = \Delta_{B/A} = (2)\left(\frac{3}{EI}\right)(4) = \frac{24}{EI} \tag{24}$$

Now it is your turn. Use the moment-area theorems to solve the following problems.

A) Determine the slope and deflection at the free end of the cantilever beam.



B) Determine the beam's deflection at B and D, and the slope of the elastic curve at C and D. The beam has a constant EI.