Structural Analysis

Lecture Series

SA54: Analysis of a Three-hinged Arch

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SA54: Analysis of a Three-hinged Arch

Several types of stress could develop in a typical beam when subjected to applied loads. For our purposes, we generally represent the state of internal stress in beams using shear force and bending moment, as shown in Figure 1.

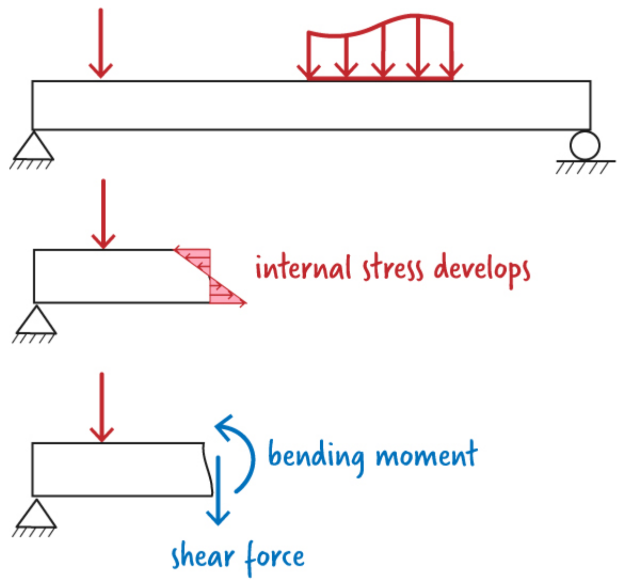

Figure 1: Internal forces in beams

In the context of structural design, there is a direct relationship between the magnitude of these forces and the size and the depth of the beam. The larger the force, especially the bending moment, the deeper the cross-section of the beam needs to be in order to carry the load safely.

For beams with a relatively long span, the bending moment could become excessively large, which would require the use of an even deeper cross-section. In situations such as these it may be desirable to curve the beam to form an arch. This configuration results in a significant reduction in bending moment, but at the expense of compressing the member (see Figure 2).

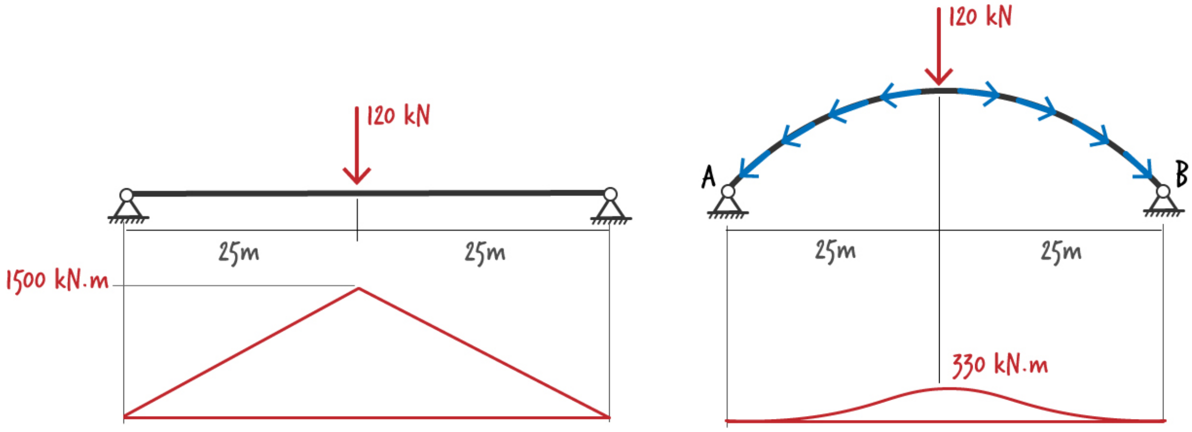

Figure 2: A beam and its comparable arch

As depicted in Figure 3, we can classify arches based on their boundary conditions. An arch could be fixed at both ends with no hinges present or we can have an arch with a hinge at its crown. You could also have two-hinged and three-hinged arches. The degree of indeterminacy of these arches varies from three to zero. With a degree of indeterminacy of zero, the three-hinged arch is considered a statically determinate system.

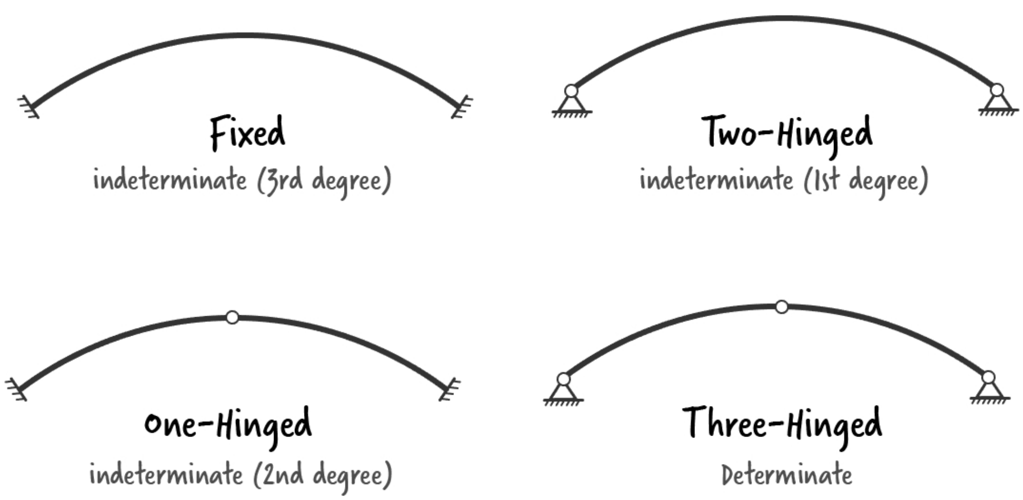

Figure 3: Types of arches

Here, we are going to focus on the analysis of a three-hinged arch. In order to analyze such a structure, we need to be able to define its shape using a mathematical function for a corresponding shape. We generally use either a circle or a parabola for this purpose.

Consider the arch shown in Figure 4. Let's refer to its height as h and use L to refer to the horizontal distance between the two supports.

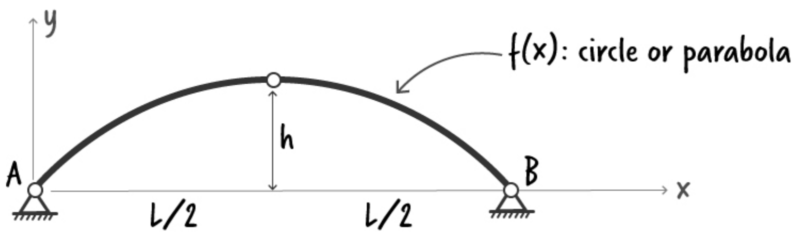

Figure 4: A parabolic arch

Suppose we wish to describe the shape of the arch using a parabolic function. We start with a general quadratic equation, like this:

$$f(x)=ax^2+bx+c \quad [1]$$

Our task is to determine the coefficients a, b and, c in terms of h and L. We know that the arch has a height of zero at the left support. So, we can write:

$$f(0)=0+0+0+c=0 \quad [2]$$

This gives us: $c=0$.

We also know that when $x=L/2$, the height of the arch is h. So, we can write:

$$f\left(\frac{L}{2}\right)=a\left(\frac{L}{2}\right)^2+b\left(\frac{L}{2}\right)=h \quad [3]$$

Further, at the right end of the arch where $x=L$, our function should evaluate to zero.

$$f(L)=aL^2+bL+0=0 \quad [4]$$

Using equations [3] and [4], we can solve for coefficients a and b to yield the following:

$$a=\frac{-4h}{L^2} \quad [5]$$

$$b = \frac{4h}{L} \qquad [6]$$

Therefore, the shape of our arch can be described using the following parabolic function:

$$f(x) = \frac{4hx}{L^2}(L - x) \qquad [7]$$

Let's now consider an arch having a height of 10 meters and it spans 50 meters in length. We wish to analyze it under a concentrated load of 120 kN placed at its crown, as shown in Figure 5.

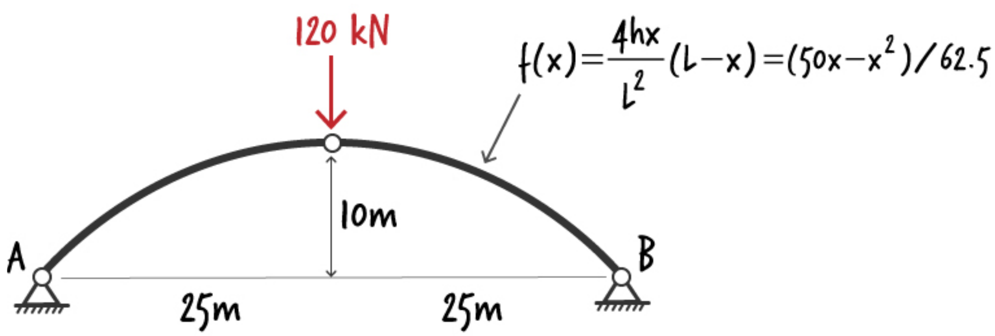

Figure 5: A parabolic arch subjected to a concentrated load

Knowing h and L, we can rewrite $f(x)$ as:

$$f(x) = \frac{50x - x^2}{62.5} \qquad [8]$$

Since the arch rests on a pin at either side, its free-body diagram involves a horizontal force and a vertical force at each end, as shown in Figure 6.

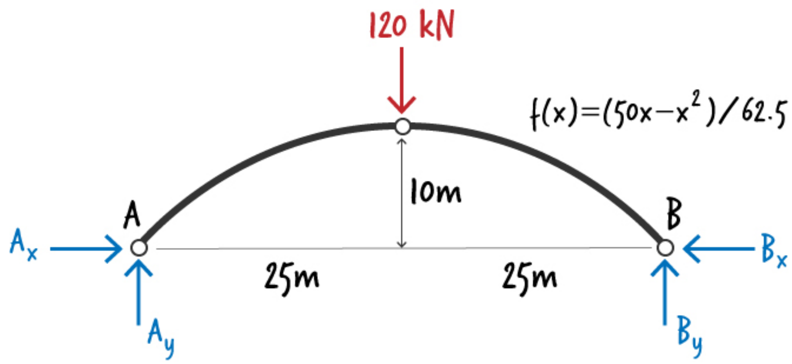

Figure 6: Free-body diagram of the arch

In this case, the two vertical reactions can be easily determined using the equilibrium equations, as shown below.

$$\sum F_y = A_y + B_y - 120 = 0 \quad [9]$$

$$\sum M_{@A} = 50B_y - 25(120) = 0 \quad [10]$$

$$A_y = B_y = 60\ kN \quad [11]$$

To determine the horizontal reactions, let's separate the left and right halves of the arch and draw the free-body diagram for each half (see Figure 7).

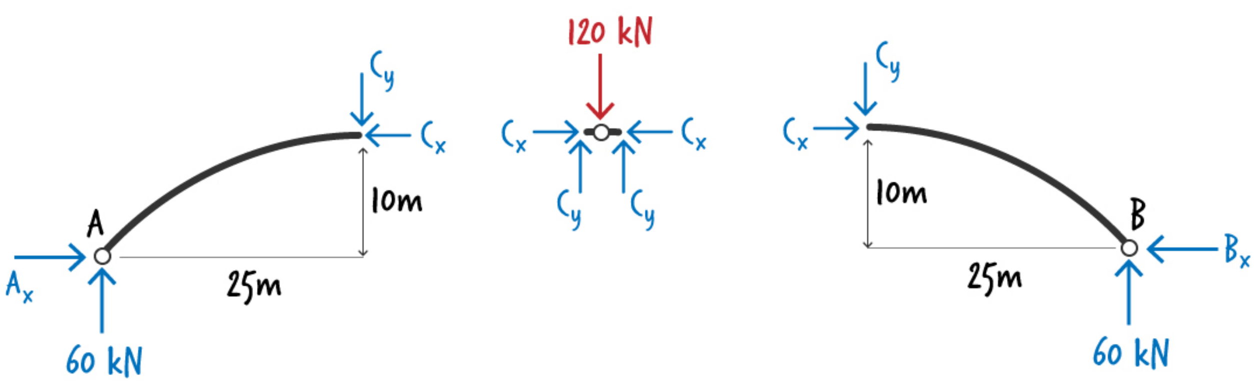

Figure 7: Free-body diagrams of the left and right arch segments

Since the bending moment at a hinge is zero, we end up with only two unknown forces, C_x and C_y, at each cut point. Also, due to the symmetrical nature of the problem, we have identical forces at the left and right sides of Point C. Now we can determine A_x using the left half of the arch. Summing the moments about the cut point, we have the following:

$$\sum M_{@C} = 25(60) - 10A_x = 0 \qquad [12]$$

We then solve the equation for A_x.

$$A_x = 150 \text{ kN} \qquad [13]$$

We can determine B_x in a similar manner, as shown below.

$$\sum M_{@C} = 25(60) - 10B_x = 0 \qquad [14]$$

$$B_x = 150 \text{ kN} \qquad [15]$$

Figure 8 shows the results of the analysis.

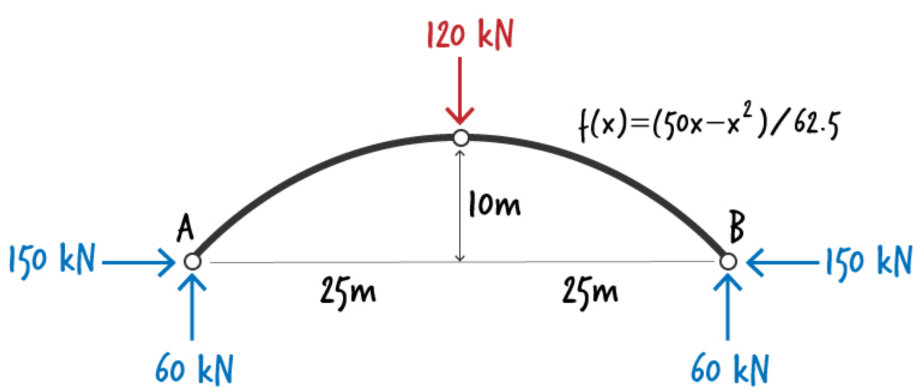

Figure 8: The arch support reactions

Suppose we are now asked to determine the internal forces in the arch including the axial force, the shear force, and the bending moment.

To find these internal forces, we will cut the arch at some distance x from the origin. The free-body diagram of the structure's left segment is shown in Figure 9. Note the horizontal and vertical distances from the origin to the cut point. We have labeled the horizontal distance x, so the vertical distance becomes $f(x)$.

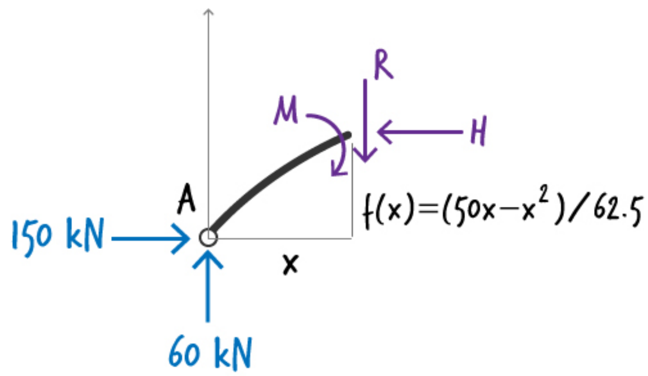

Figure 9: The arch's internal forces shown in the global coordinate system

The free-body diagram involves the three unknown forces M, H, and R. We can determine M by writing the sum of the moments about the cut point as shown in Equation [16].

$$M+60x-\frac{150(50x-x^2)}{62.5}=0 \qquad [16]$$

Solving the equation for M, we get:

$$M=60x-2.4x^2 \qquad [17]$$

As the above equation suggests, the bending moment in the arch varies as a function of x in a nonlinear manner. Furthermore, since the sum of the forces in the x-direction must be zero, H must be 150 kN and R must be 60 kN in order for the sum of the forces in the y-direction to be zero.

However, note that H is NOT the axial force and R is NOT the shear force in the member. The axial force must be in the tangential direction at x, and the shear force must be in the radial direction, as shown in Figure 10.

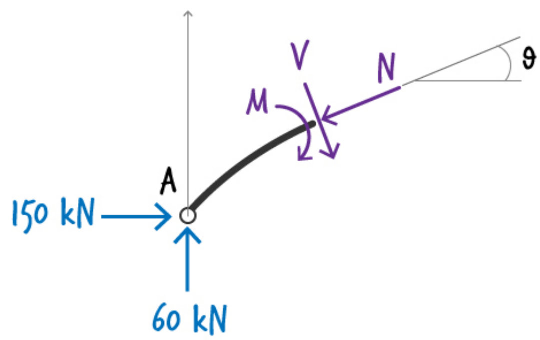

Figure 10: Internal forces in the arch in the local coordinate system

If we refer to the angle that the tangent to the curve makes with the horizontal axis as ϑ , then the tangent of the angle can be expressed in terms of the derivative of $f(x)$ with respect to x .

$$\tan(\vartheta)=\frac{df}{dx}=0.8-0.032x \tag{18}$$

Knowing the tangent of an angle, we can determine the angle itself. We can now express N and V in terms of H , R , and ϑ as shown in Equations [19] and [20].

$$N=H\cos\vartheta+R\sin\vartheta \tag{19}$$

$$V=R\cos\vartheta-H\sin\vartheta \tag{20}$$

Since H is 150 kN and R is 60 kN, Equations [19] and [20] can be rewritten as the following:

$$N=150\cos\vartheta+60\sin\vartheta \tag{21}$$

$$V=60\cos\vartheta-150\sin\vartheta \tag{22}$$

Now, let's use Equations [17], [21], and [22] to draw the moment, thrust, and shear diagrams, respectively.

To draw the moment diagram, we need to graph Equation [17]. This gives us a diagram for the left half of the arch, as shown in Figure 11. Note that since the geometry and the load are symmetrical about the centerline of the arch, the diagram for the right half of the structure will be identical to that of the left half.

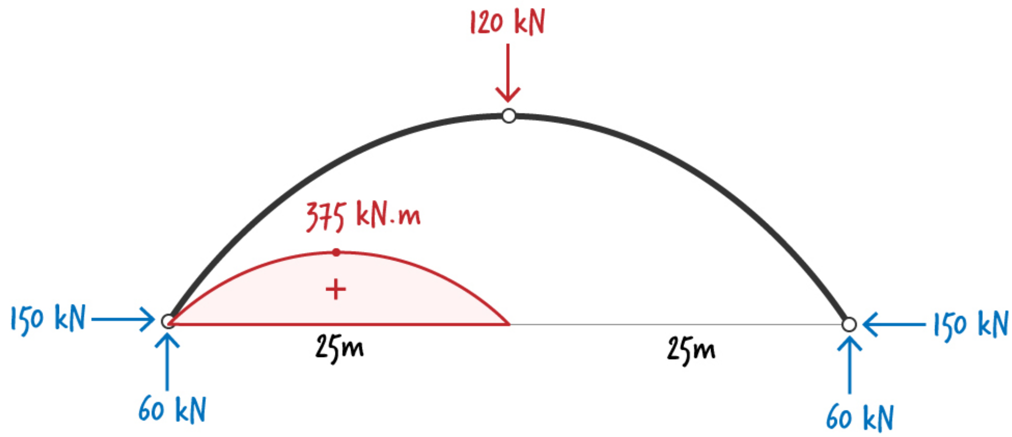

Figure 11: Moment diagram for the left half of the arch

Another item to note is that the bending moment at the hinge at either end of the segment is zero. We can verify this by evaluating Equation [17] at zero and 25.

We can also determine the point at which the maximum moment occurs by setting the derivative of the moment equation to zero, and then solving for x, as shown in Equation [23].

$$\frac{dM}{dx}=60-4.8x=0 \tag{23}$$

Solving the above equation for x, we get: $x=12.5$ meters.

The equation tells us that the maximum moment occurs 12.5 meters to the right of A. Therefore, according to Equation [17], the magnitude of the maximum moment equals 375 kNm as shown in Equation [24].

$$M(12.5)=60(12.5)-2.4(12.5)^2=375\ kN.m \tag{24}$$

The complete moment diagram for the arch is given in Figure 12.

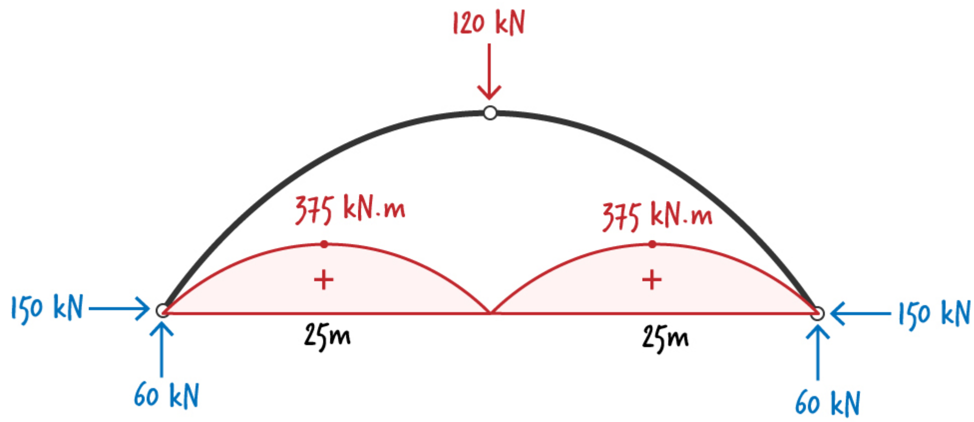

Figure 12: Moment diagram for the entire arch

To draw the diagram for the axial force, also called the thrust diagram, we need to graph Equation [21]. We have already determined an algebraic expression for tangent θ (see Equation [18]). Using the trigonometric properties of an angle, we can then express the sine and cosine of the angle in terms of its tangent, as shown in Equations [25] and [26].

$$\cos(\theta)=\frac{1}{\sqrt{1+\tan^2(\theta)}} \Rightarrow \frac{1}{\sqrt{1+(0.8-0.032x)^2}} \quad [25]$$

$$\sin(\theta)=\tan(\theta)\cos(\theta) \Rightarrow \frac{0.8-0.032x}{\sqrt{1+(0.8-0.032x)^2}} \quad [26]$$

Substituting [25] and [26] into Equation [21], we get the following:

$$N=\frac{198-1.92x}{\sqrt{1+(0.8-0.032x)^2}} \quad [27]$$

The graph of this equation is depicted in Figure 13.

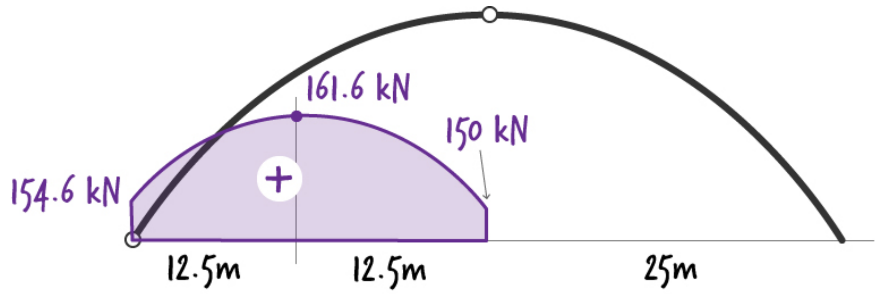

Figure 13: Thrust diagram for the left half of the arch

To determine the maximum axial force in the segment, we set the derivative of N to zero, as shown in Equation [28].

$$\frac{dN}{dx}=0\Rightarrow 1875-150x=0 \tag{28}$$

Solving the above equation for x, we get the following:

$$x=12.5 \tag{29}$$

Therefore, the maximum axial force in the arch occurs 12.5 meters from the left support. The magnitude of the force is 161.6 kN when evaluated using Equation [27].

$$N(12.5)=\frac{198-1.92(12.5)}{\sqrt{1+(0.8-0.032(12.5))^2}}=161.6\text{ kN} \tag{30}$$

Again, the diagram for the right half of the arch is identical to that of the left half, so the complete thrust diagram for the arch can be constructed as shown below in Figure 14.

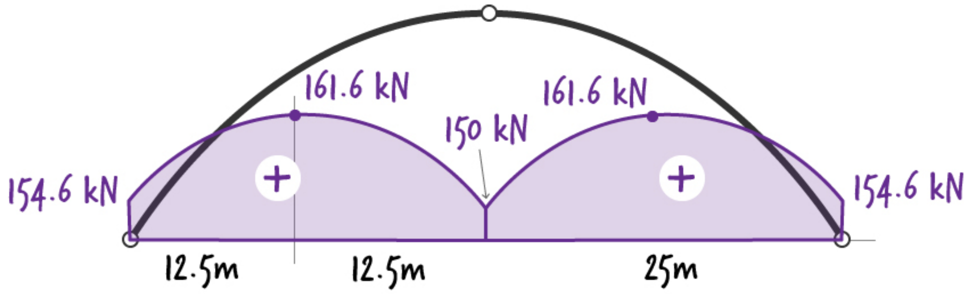

Figure 14: Thrust diagram for the entire arch

Finally, to draw the shear diagram, we can use Equation [22].

Using Equations [25] and [26], we can simplify Equation [22] as follows:

$$V=\frac{-60+4.8x}{\sqrt{1+(0.8-0.032x)^2}} \quad [31]$$

Note the numerator of the equation which tells us that the shear is zero when $x=12.5$. The graph of the equation is shown in Figure 15.

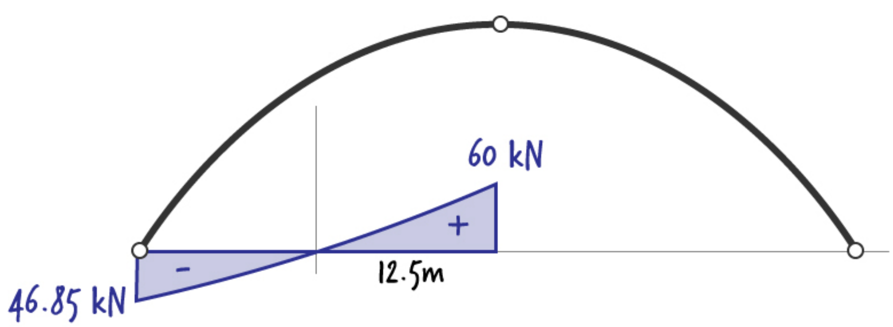

Figure 15: Shear diagram for the left half of the arch

The shear is negative 46.85 kN at the left end of the left side segment of the arch. The shear is positive 60 kN at the right end of this particular segment. The complete shear diagram is shown in Figure 16.

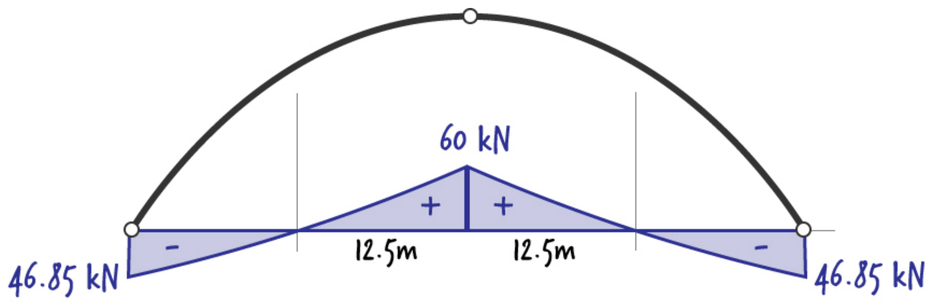

Figure 16: Shear diagram for the entire arch

Figure 17 shows the moment diagram, thrust diagram, and shear diagram for the entire arch.

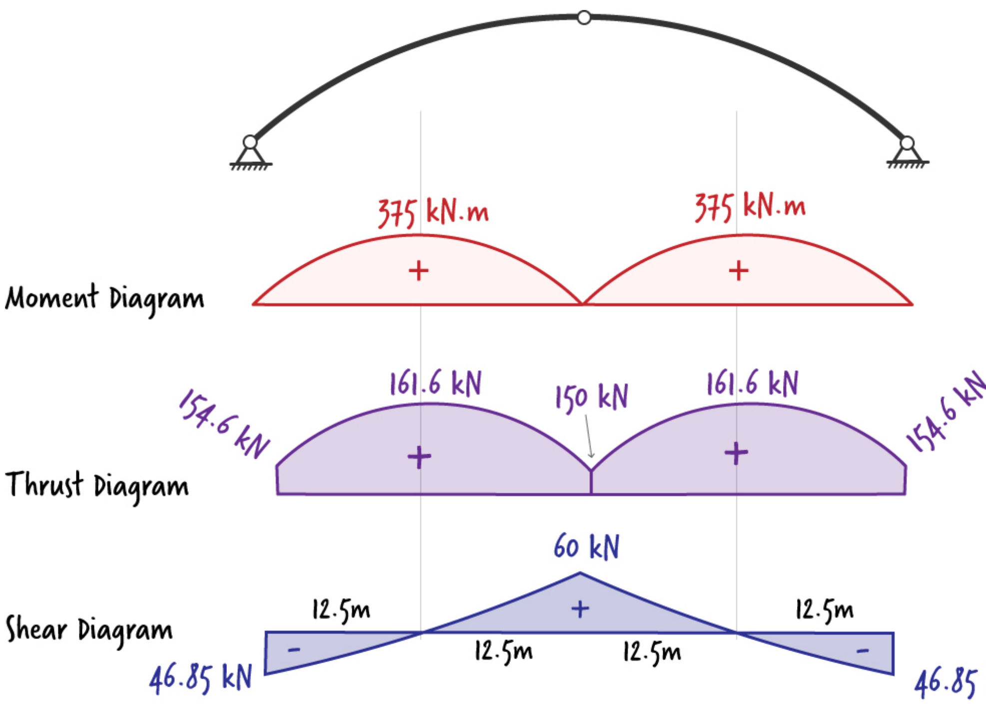

Figure 17: Moment, thrust, and shear diagrams for the arch