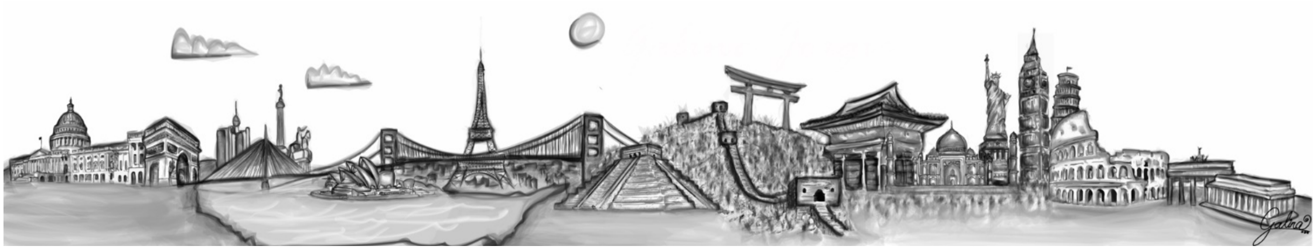
Structural Analysis

Lecture Series



SA55: Analysis of a Three-hinged Arch Bridge

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2018

Structural Analysis – SA55

Analysis of a Three-hinged Arch Bridge

Prerequisite: Lecture SA54

In this lecture, we are going to show how to determine the maximum bending moment in a three-hinged arch due to moving loads.

Consider the structure shown in Figure 1. We wish to determine the locations and magnitudes of the absolute maximum positive and negative moments in the arch.

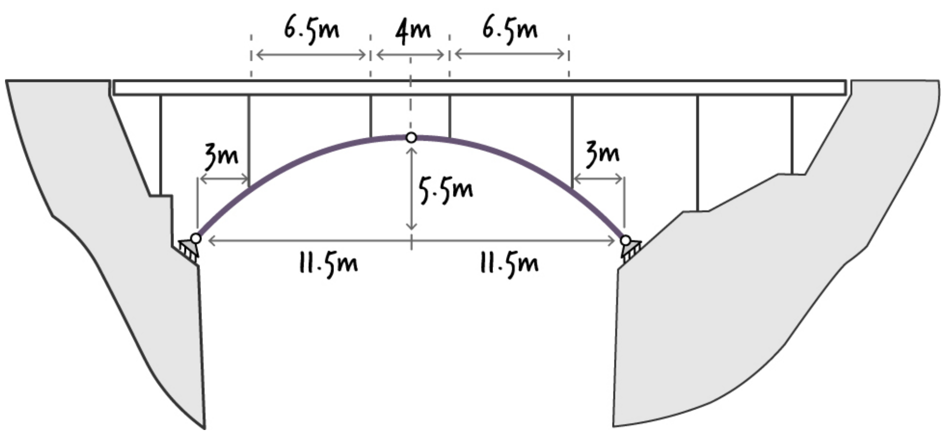

Figure 1: A three-hinged arch bridge.

As depicted in Figures 1 and 2, moving loads are transmitted from the bridge deck to the supporting arch through four vertical posts. At any point in time, one or more posts may be involved in transferring the loads to the supporting sub-structure. For example, when two vehicles are positioned on the bridge as shown in Figure 2, their loads are exerted on the arch at the circled points.

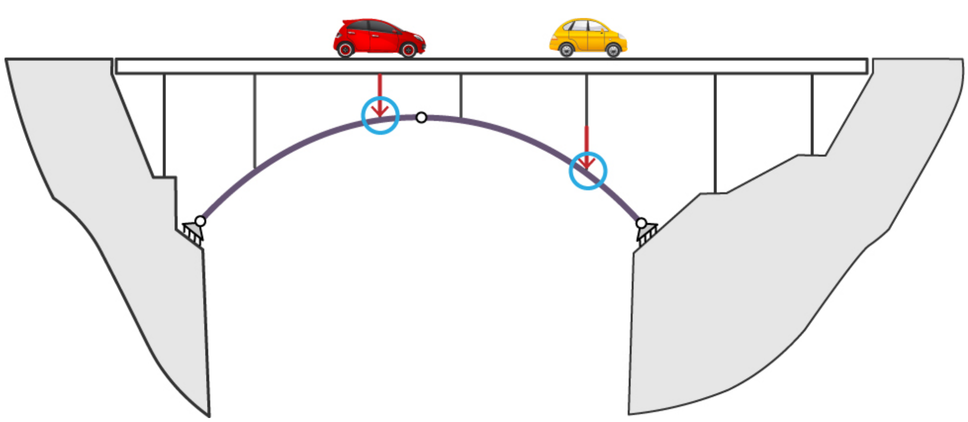

Figure 2: Load distribution on a three-hinged arch.

Suppose the maximum vehicular load permitted on our bridge is referred to as P_{max}. The arch can be subjected to four such loads as shown in Figure 3. For convenience, we are going to analyze the arch using unit loads instead of the maximum load. We will then multiply the results by P_{max} in order to calculate the correct moment values.

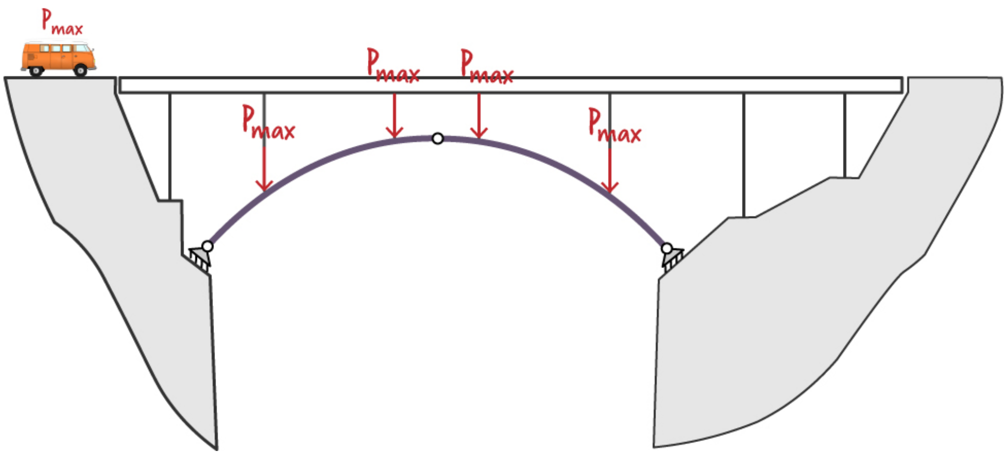

Figure 3: Maximum loads exerted on the bridge.

It is important to note that all four loads may not have to be present at the same time for the maximum moment to develop. In fact, as will be seen later, the maximum positive moment develops when only one of the posts is loaded. And for the maximum negative moment, only three of the four posts need to be loaded.

To determine the locations and magnitudes of the critical moments, we are going to employ the principle of superposition. First, we analyze the arch under the four separate loading cases as shown in Figure 4. Then, we calculate the maximum moment values by combining the results obtained from the individual cases.

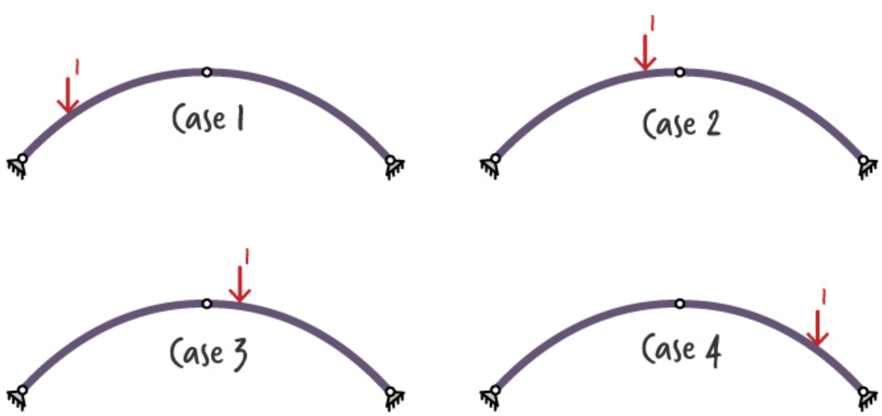

Fig. 4: Four loading cases.

Case 1: Unit load at the left-most post

For this load case, we are going to analyze the arch under a vertical unit load placed three meters away from the left pin support where a vertical post transfers the load from the bridge deck to the arch, as shown in Figure 5.

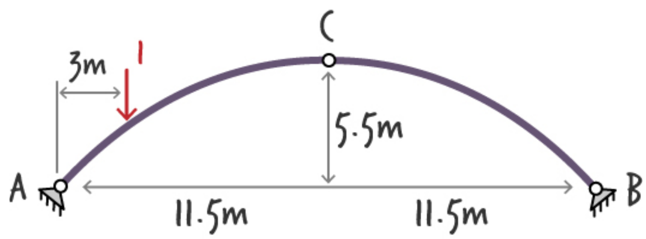

Figure 5: Load case 1.

The governing equation describing the shape of the arch is:

$$f(x) = \frac{4hx}{L^2}(L - x) \qquad [1]$$

Where h is the height of the arch and L is the horizontal distance between the two supports. Please see Lecture SA54 for a derivation of this equation.

The arch has a height of 5.5 meters, and it spans 23 meters, so we can write:

$$f(x) = \frac{4(5.5)x}{23^2}(23 - x) = \frac{23x - x^2}{24} \qquad [2]$$

Given the pin supports at A and B, the free body diagram for the entire arch can be drawn as shown in Figure 6.

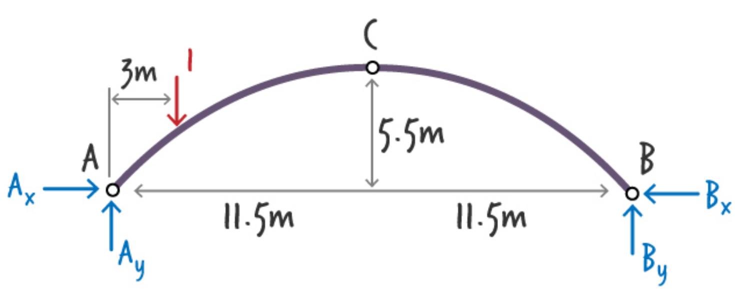

Figure 6: Free-body diagram of the arch (case 1).

Using the static equilibrium equations, the two vertical support reactions can be determined as follows:

$$\sum_{@A} M = 3(1) - 23B_y = 0 \quad [3]$$

$$\sum F_y = A_y + B_y - 1 = 0 \quad [4]$$

$$B_y = 0.13 \;\; kN \quad [5]$$

$$A_y = 0.87 \;\; kN \quad [6]$$

Now, let’s separate the arch into two segments at point C, as shown in Figure 7.

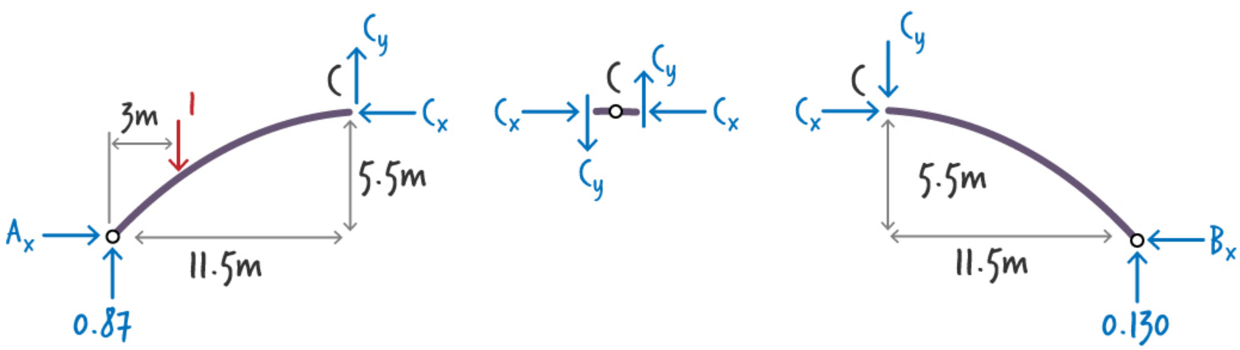

Figure 7: Free-body diagram of the separated arch (case 1).

Note that since there is a hinge at C, no bending moment develops at the crown of the arch.

Using the free-body diagram of the left segment, we can determine the horizontal support reaction at A by setting the sum of moments about C to zero, and then solving the equation for A_x.

$$\sum_{@C} M = 0.87(11.5) - 5.5A_x - 1(11.5 - 3) = 0 \quad [7]$$

$$A_x = 0.27 \;\; kN \quad [8]$$

Similarly, the free-body diagram for the right segment of the arch can be used for calculating B_x.

$$\sum_{@C} M = 5.5B_x - 11.5(0.13) = 0 \quad [9]$$

$$B_x = 0.27 \;\; kN \quad [10]$$

Knowing the support reactions at A and B, we are now ready to write the moment equation for the entire arch.

According to Figure 6, we need to write two moment equations; one equation for the segment of the arch to the left of the unit load, and another equation for the segment to the right of the load.

To write the first moment equation, we cut the segment at an arbitrary point to the left of the unit load, and then draw the free-body diagram of the left segment, as shown in Figure 8.

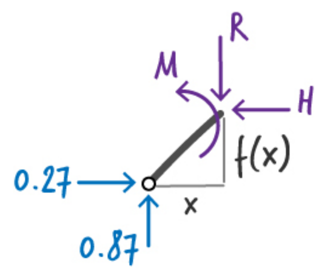

Figure 8: Free-body diagram of the first segment of the arch for load case 1.

Three unknown forces are present in this diagram. However, since we are only interested in determining the bending moment equation for the arch, we are going to ignore the horizontal force (H) and the vertical force (R). Taking the sum of the moments about the cut point, we get the following equation.

$$M + 0.27 f(x) - 0.87x = 0 \qquad [11]$$

Substituting Equation [2] in the above equation, we get:

$$M + 0.27\left(\frac{23x - x^2}{24}\right) - 0.87x = 0 \qquad [12]$$

Which can then be written in the simplified form as:

$$M = 0.011x^2 + 0.61x \qquad 0 \le x \le 3 \qquad [13]$$

Note that the above equation is valid only for the range from 0 to 3.

For the second equation, we are going to cut the arch to the right of the unit load (see Figure 6), which gives us the following free-body diagram.

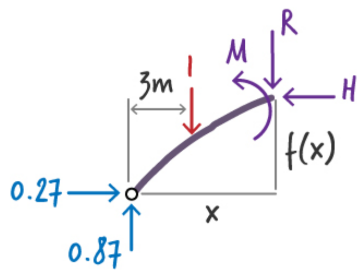

Figure 9: Free-body diagram of the second segment of the arch for load case 1.

Similar to the first moment equation, we simply need to set the sum of the moments about the cut point to zero in order to determine the moment equation for the right segment of the arch.

$$M + 0.27 f(x) - 0.87x + (x - 3) = 0 \qquad [14]$$

Substituting Equation [2] in the above equation, we get:

$$M + 0.27(\frac{23x - x^2}{24}) - 0.87x + x - 3 = 0 \qquad [15]$$

In a simplified form, the above equation can be written as:

$$M = 0.11x^2 - 0.39x + 3 \qquad 3 \le x \le 23 \qquad [16]$$

Let us draw the moment diagram for load case 1 by graphing Equations [13] and [16].

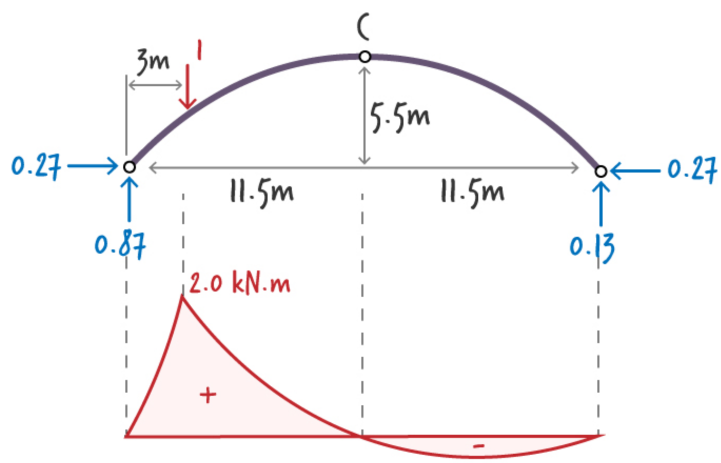

Figure 10: Moment diagram for load case 1.

Note that the bending moment reaches its maximum positive value under a unit load which is exerted on the arch through the left-most post in the bridge.

Case 2: Unit load at the left inner post

For the second loading case, we placed the unit load 9.5 meters away from point A, as depicted in Figure 11.

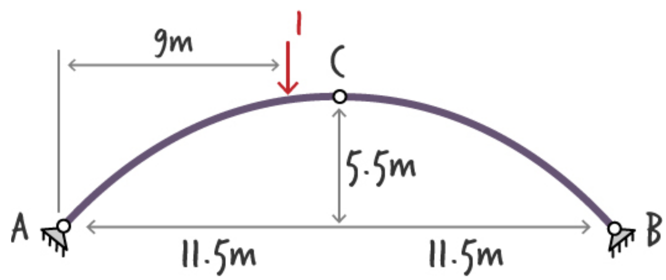

Figure 11: Load case 2.

The free-body diagram of the arch is shown below.

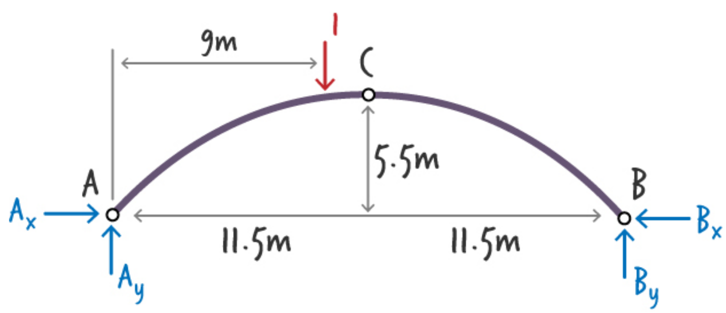

Figure 12: Free-body diagram of the arch (case 2).

We calculate the vertical support reactions using the equilibrium equations, as shown below.

$$\sum_{@A} M = 23B_y - 9.5(1) = 0 \quad [17]$$

$$\sum F_y = A_y + B_y - 1 = 0 \quad [18]$$

$$B_y = 0.41 \ kN \quad [19]$$

$$A_y = 0.59 \ kN \quad [20]$$

To determine the horizontal support reactions, we first separate the arch into two segments as shown in Figure 13.

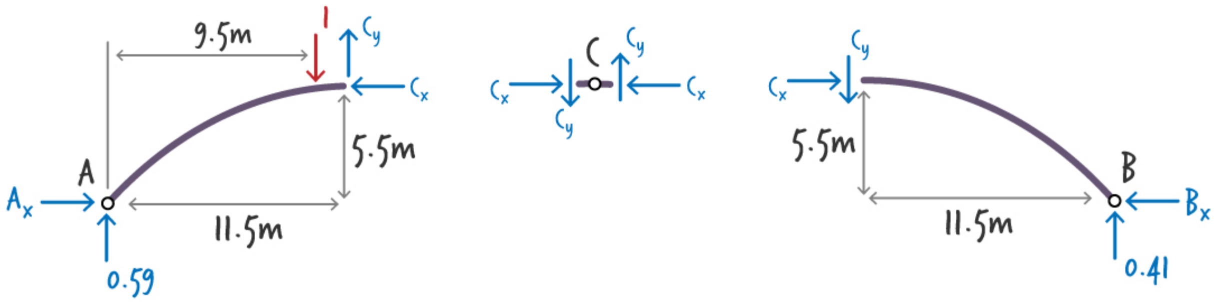

Figure 13: Free-body diagram of the separated arch (case 2).

Then, using the left free-body diagram, we determined A_x, and using the right segment, we can now calculate B_x.

$$\sum_{@C} M = 0.59(11.5) - 5.5A_x - (11.5 - 9.5) = 0 \quad [21]$$

$$A_x = 0.86 \;\; kN \quad [22]$$

$$\sum_{@C} M = 0.41(11.5) - 5.5B_x = 0 \quad [23]$$

$$B_x = 0.86 \;\; kN \quad [20]$$

Similar to Case 1, we need to write two moment equations for this loading case. We are going to cut the arch just to the left of the unit load for the first equation and just to the right of the unit load for the second equation. Figure 14 shows the two resulting free-body diagrams.

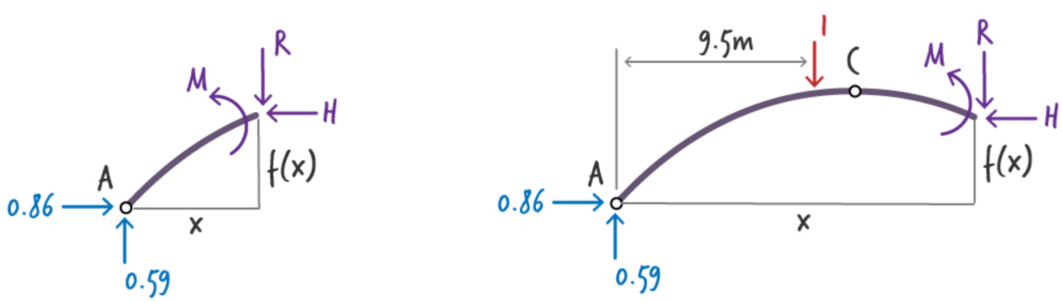

Figure 14: Free-body diagrams of the two segments of the arch for load case 2.

For the first diagram, we can write:

$$M + 0.86f(x) - 0.59x = 0 \qquad [25]$$

Or,

$$M + 0.86\left(\frac{23x - x^2}{24}\right) - 0.59x = 0 \qquad [26]$$

In simplified form, the above equation can be written as:

$$M = 0.036x^2 - 0.24x \qquad 0 \le x \le 9.5 \qquad [27]$$

And for the second segment where x is between 9.5 and 23 meters, we can write:

$$M + 0.86f(x) - 0.59x + (x - 9.5) = 0 \qquad [28]$$

Or,

$$M + 0.86\left(\frac{23x - x^2}{24}\right) - 0.59x + x - 9.5 = 0 \qquad [29]$$

Or,

$$M = 0.036x^2 - 1.24x + 9.5 \qquad 9.5 \le x \le 23 \qquad [30]$$

Now, using Equations [27] and [30], the moment diagram for the arch can be drawn as shown in Figure 15.

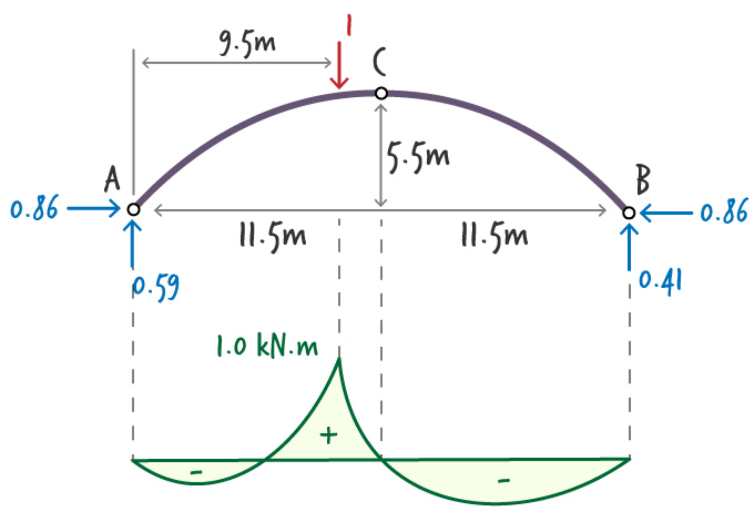

Figure 15: Moment diagram for the case 2.

Case 3: Unit load at the right inner post

Here, the load is placed two meters past the hinge at C, as shown in Figure 16.

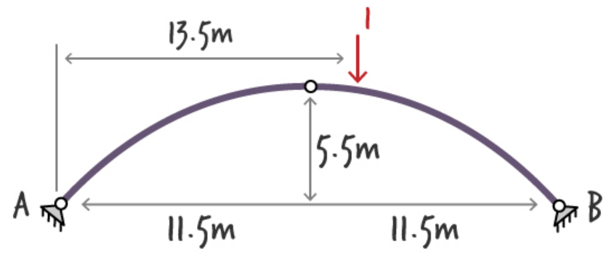

Figure 16: Load case 3.

The same process will be repeated for loading case 3. First, we draw the free-body diagram for the arch (see Figure 17), and then we calculate the vertical support reactions at A and B.

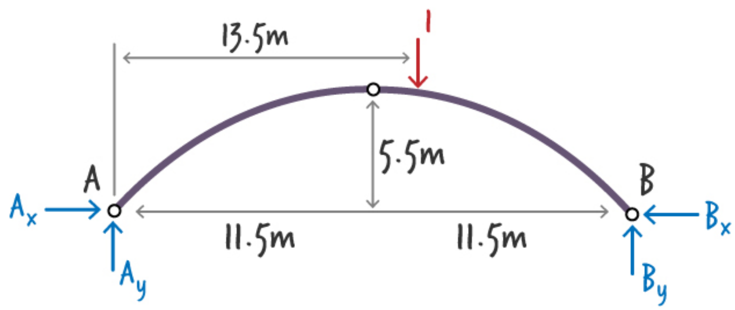

Figure 17: Free-body diagram of the arch (case 3).

$$\sum_{@A} M = 23B_y - 13.5(1) = 0 \qquad [31]$$

$$\sum F_y = A_y + B_y - 1 = 0 \qquad [32]$$

$$B_y = 0.59 \;\; kN \qquad [33]$$

$$A_y = 0.41 \;\; kN \qquad [34]$$

Now, we determine the horizontal reactions at A and B, first by splitting the arch at C (see Figure 18), and then by writing and solving the equilibrium equations.

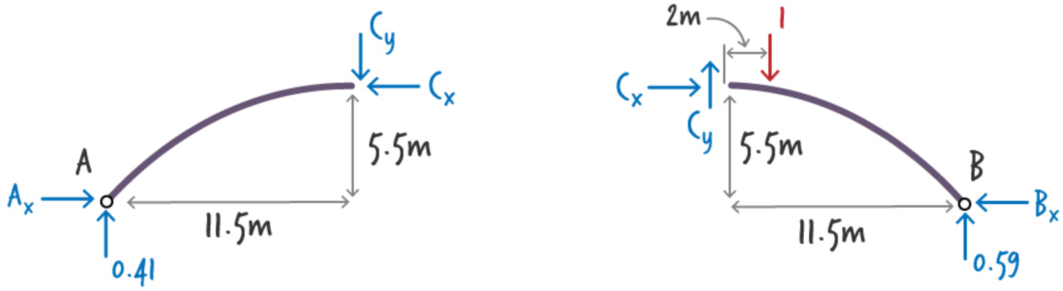

Figure 18: Free-body diagram of the separated arch (case 3).

$$\sum_{@C} M = 0.41(11.5) - 5.5A_x = 0 \quad [35]$$

$$A_x = 0.86 \quad [36]$$

$$\sum_{@C} M = 0.59(11.5) - 5.5B_x - 1(2) = 0 \quad [37]$$

$$B_x = 0.86 \quad [38]$$

To write the moment equations for this loading case, we cut the arch to the left of the unit load, and we cut it to the right of the unit load. The two resulting free-body diagrams are shown in Figure 19.

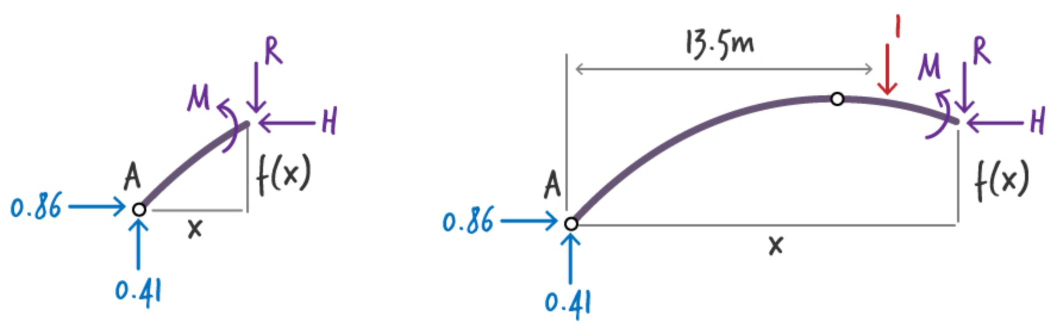

Figure 19: Free-body diagrams of the left and right segments of the arch for load case 3.

For the first segment where x is between 0 and 13.5 meters, the moment equation can be written as:

$$M - 0.41x + 0.86f(x) = 0 \quad [39]$$

Substituting Equation [2] in [39], we get:

$$M - 0.41x + 0.86(\frac{23x - x^2}{24}) = 0 \qquad [40]$$

Or,

$$M = 0.036x^2 - 0.41x \qquad 0 \le x \le 13.5 \qquad [41]$$

And for the second segment where x is between 13.5 and 23 meters, we can write:

$$M - 0.41x + 0.86 f(x) + 1(x - 13.5) = 0 \qquad [42]$$

Or,

$$M - 0.41x + 0.86(\frac{23x - x^2}{24}) + x - 13.5 = 0 \qquad [43]$$

Which in simplified form becomes:

$$M = 0.036x^2 - 1.41x + 13.5 \qquad 13.5 \le x \le 23 \qquad [44]$$

Then, the bending moment diagram for the entire arch can be drawn using Equations [41] and [44], as shown in Figure 20.

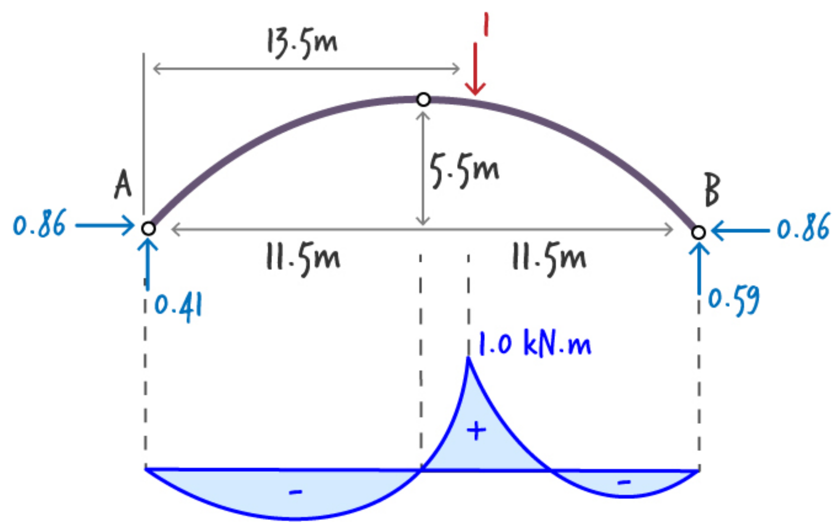

Figure 20: Moment diagram for load case 3.

Case 4: Unit load at the right-most post

In the last loading case, the unit load is placed 3 meters to the left of B (see Figure 21).

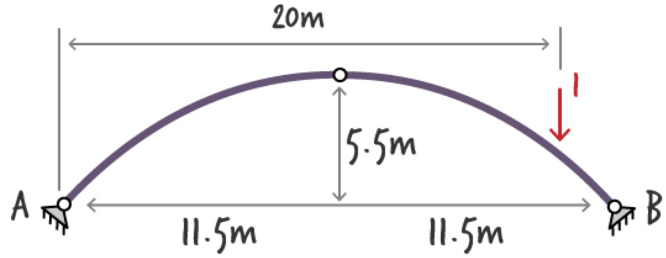

Figure 21: Load case 4.

This results in the following free-body diagram:

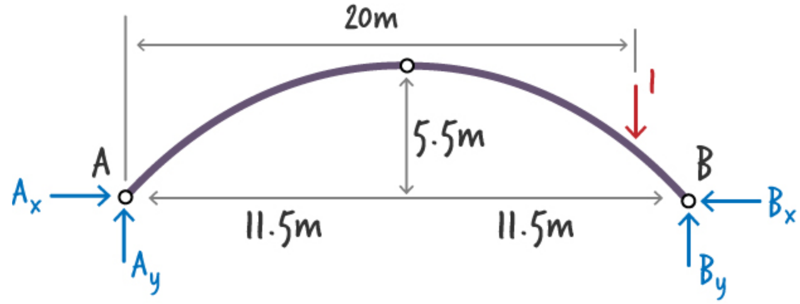

Figure 22: Free-body diagram of the arch (case 4).

The vertical support reactions at A and B are computed as follows:

$$\sum_{@A} M = 23B_y - 20(1) = 0 \quad [45]$$

$$\sum F_y = A_y + B_y - 1 = 0 \quad [46]$$

$$B_y = 0.87 \ \ kN \quad [47]$$

$$A_y = 0.13 \ \ kN \quad [48]$$

Similar to the previous loading case, we separate the arch into two segments at C (see Figure 23).

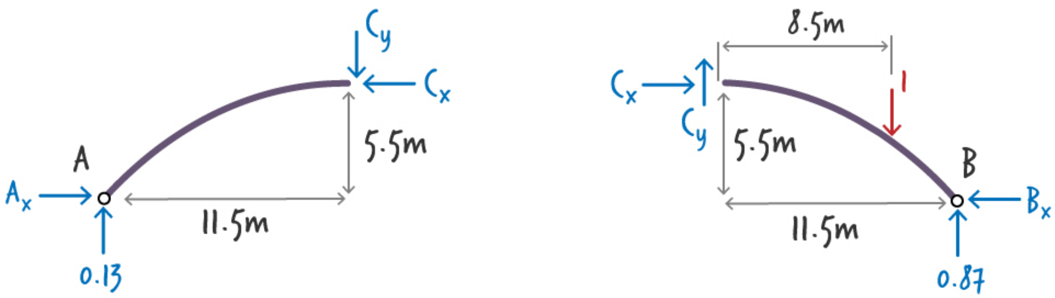

Figure 23: Free-body diagram of the separated arch (case 4).

We then write, and solve, the moment equilibrium equation at C for each free-body diagram in order to determine the two horizontal reactions.

Horizontal reactions at A and B are found by first splitting the arch at C and then writing and solving the equilibrium equations, like this:

$$\sum_{@C} M = 0.13(11.5) - 5.5A_x = 0 \quad [49]$$

$$A_x = 0.27 \;\; kN \quad [50]$$

$$\sum_{@C} M = 0.87(11.5) - 5.5B_x - 1(8.5) = 0 \quad [51]$$

$$B_x = 0.27 \;\; kN \quad [52]$$

To determine the moment equations for the entire arch, we need to cut it to the left and to the right of the unit load (see Figure 24).

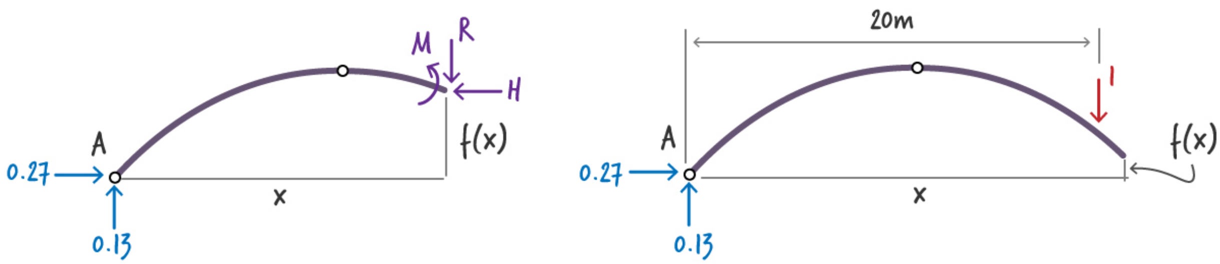

Figure 24: Free-body diagram of the separated arch (case 4).

The two moment equations are written in the following manner:

$$M - 0.13x + 0.27f(x) = 0 \quad [53]$$

$$M - 0.13x + 0.27\left(\frac{23x - x^2}{24}\right) = 0 \quad [54]$$

$$M = 0.011x^2 - 0.13x \qquad 0 \le x \le 20 \tag{55}$$

$$M - 0.13x + 0.27f(x) + 1(x-20) = 0 \tag{56}$$

$$M - 0.13x + 0.27\left(\frac{23x - x^2}{24}\right) + x - 20 = 0 \tag{57}$$

$$M = 0.011x^2 - 1.13x + 20 \qquad 20 \le x \le 23 \tag{58}$$

The bending moment diagram for the entire arch for this loading case is shown in Figure 25.

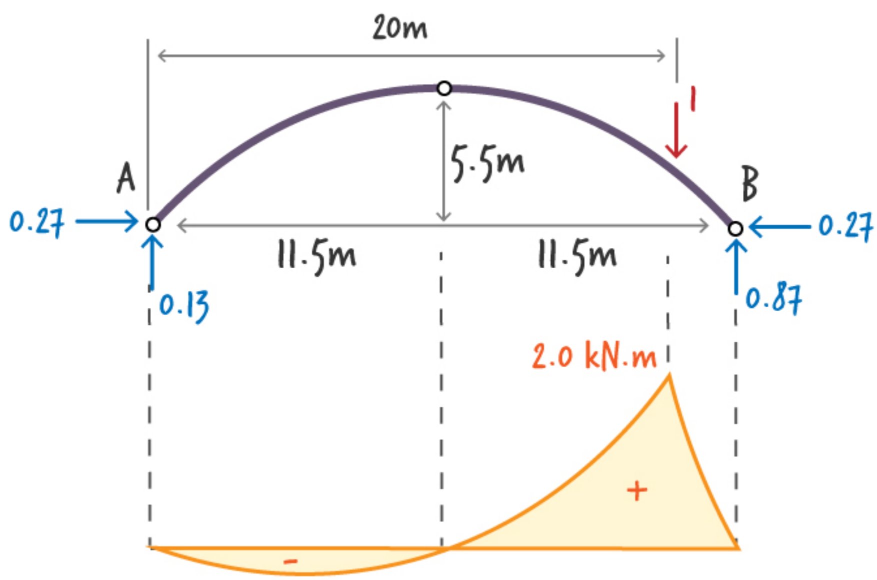

Figure 25: Moment diagram for load case 4.

Determining the absolute maximum positive and negative moments

Figure 26 shows the four moment diagrams side-by-side. A visual inspection of the diagram reveals that the maximum positive moment takes place 3 meters away from the left support under loading case 1 when no other load is present. The same maximum moment develops three meters to the left of the support at B under loading case 4 when no other load is present.

So, the absolute maximum positive moment develops at two points in the arch when a single vehicle is on the bridge. When the vehicle, having a weight of P_{max}, is at the top of the left-most post, a maximum moment of $2P_{max}$ develops at the post. When the vehicle reaches the right-most post, the moment in the arch at that location also reached $2P_{max}$. No other point in the arch under any other loading scenario could have a moment larger than $2P_{max}$.

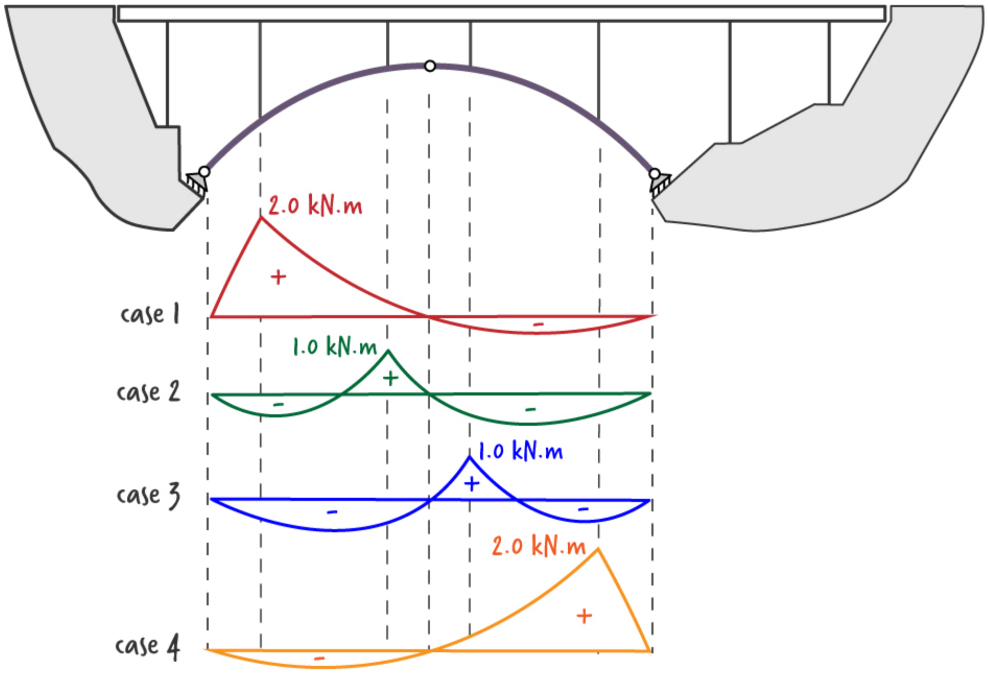

Figure 26: Moment diagrams of the four cases.

A close examination of Figure 26 reveals that the maximum negative moment develops in the left region of the arch when loading cases 2, 3 and 4 are present and loading case 1 is absent. Such a loading scenario results in adding three negative moment values (for loading cases 2, 3 and 4) without the need to subtract the positive value, due to loading case 1, from the total.

The moment expression can be obtained by writing the algebraic sum of Equations [27], [41] and [55]. The resulting equation is:

$$M = 0.083x^2 - 0.78x \quad [59]$$

To determine the location of the maximum negative moment, we set the derivative of the equation to zero and solve for x.

$$\frac{dM}{dx} = 2(0.083)x - 0.78 = 0 \quad [60]$$

$$x = 4.7\ m \quad [61]$$

This means, the absolute maximum negative moment takes place 4.7 meters to the right of A. To determine the magnitude of the moment, first evaluate Equation [59] at x = 4.7. This yields a negative 1.85. Then we multiply this value by P_{max} to get the actual moment value due to the vehicular load.

So, the absolute maximum negative moment is:

$$-1.85P_{\max}$$

The moment reaches this value 4.7 meters away from A when three vehicles are on the bridge; one on top of the inner left post, another on top of the inner right post, and the third one on top of the right-most post.

A close examination of Figure 26 also reveals that moment could reach its maximum negative value in the right segment of the arch only when loading cases 1, 2 and 3 are present.

The expression for this combined moment can be obtained by adding Equations [16], [30] and [44]. The resulting equation is:

$$M = 0.083x^2 - 3.05x + 26 \qquad [62]$$

At the point of maximum moment, the derivative of this equation must vanish. Setting the derivative to zero and solving for x, we get:

$$\frac{dM}{dx} = 2(0.083)x - 3.05 = 0 \qquad [63]$$

$$x = 18.3\ m \qquad [64]$$

So, the absolute maximum moment could also develop 18.3 meters away from A. Substituting 18.3 for x in Equation [62], we get negative 1.85. This is the maximum moment due to unit loads. We multiply it by P_{max} to get the actual moment value. Then, we can conclude that absolute maximum moment could also develop in the right segment of the arch when three vehicles are on the bridge; one on top of the left-most post, one on top of the left inner post, and the other on top of the right inner post. Figure 27 shows the summary of the results.

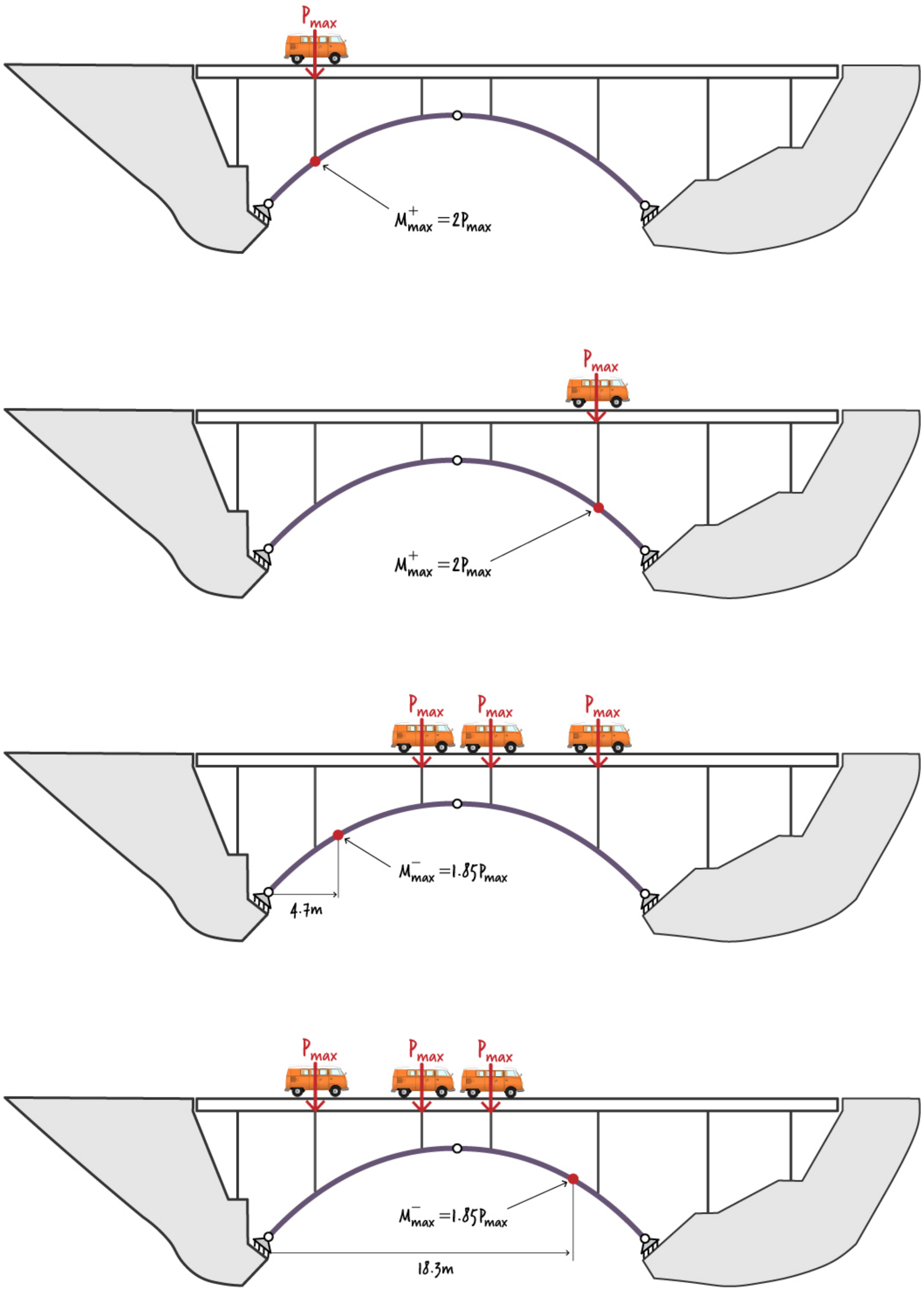

Figure 27: Summary of the results.